CHAPTER III

Congruent Triangles

1. Understanding the Meaning of Congruent Polygons

In modern industry, it is often necessary to make many copies of a part so that each copy will have the same size and shape. For example, a machine can stamp out many duplicates of a piece of metal, each copy having the same size and shape as the original.

One way to discover whether or not two polygons have the same size and shape is to place one polygon upon the other. If the figures can be turned in such a way that the sides of one polygon *fit exactly* upon the sides of the other and the angles of one polygon *fit exactly* upon the angles of the other, we say that the polygons *coincide*. Polygons that can be made to coincide are called *congruent polygons*.

For example, we can make the two polygons shown in Fig. 3–1 coincide if we make vertex E correspond to vertex A, vertex F correspond to vertex B, vertex G correspond to vertex C, and vertex H correspond to vertex D. Therefore, we would say that "polygon $ABCD$ is congruent to polygon $EFGH$," symbolized by "polygon $ABCD \cong$ polygon $EFGH$." Notice that when we named congruent polygons, the order in which we wrote their vertices indicates the one-to-one correspondence that was set up between the vertices of polygon $ABCD$ and the vertices of polygon $EFGH$.

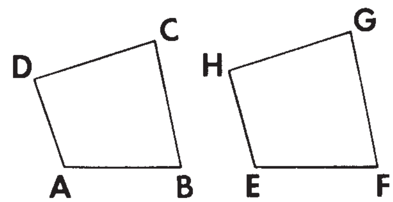


Fig. 3–1

A corresponds to E, and E corresponds to A. B corresponds to F, and F corresponds to B. C corresponds to G, and G corresponds to C. D corresponds to H, and H corresponds to D. (See Fig. 3–2.)

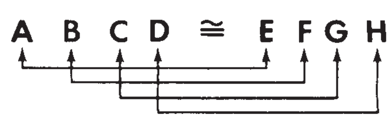


Fig. 3–2

In the future, we will name two polygons that are congruent so as to indicate the correspondences between the vertices of the polygons.

Corresponding Angles

Observe that in congruent polygons $ABCD$ and $EFGH$ (Fig. 3–1), since vertex A corresponds to vertex E, then angle E would fit exactly on angle A. Hence, $\angle A \cong \angle E$. Angles A and E are called *corresponding angles* because their vertices correspond in the one-to-one correspondence that was set up between the vertices of polygon $ABCD$ and polygon $EFGH$.

Similarly, since vertex B corresponds to vertex F, angles B and F are corresponding angles and $\angle B \cong \angle F$. Since vertex C corresponds to vertex G, angles C and G are corresponding angles and $\angle C \cong \angle G$. Since vertex D corresponds to vertex H, angles D and H are corresponding angles and $\angle D \cong \angle H$.

Hence, in these congruent polygons, *all the pairs of corresponding angles are congruent.*

Corresponding Sides

Observe that in congruent polygons $ABCD$ and $EFGH$ (Fig. 3–1), since vertex A corresponds to vertex E, and vertex B corresponds to vertex F, side $\overline{AB}$ would fit exactly on side $\overline{EF}$. Hence, $\overline{AB} \cong \overline{EF}$. Sides $\overline{AB}$ and $\overline{EF}$ are called *corresponding sides* because their endpoints correspond in the one-to-one correspondence that was set up between the vertices of polygon $ABCD$ and the vertices of polygon $EFGH$.

Similarly, since vertex B corresponds to vertex F, and vertex C corresponds to vertex G, sides $\overline{BC}$ and $\overline{FG}$ are corresponding sides and $\overline{BC} \cong \overline{FG}$. Since vertex C corresponds to vertex G, and vertex D corresponds to vertex H, sides $\overline{CD}$ and $\overline{GH}$ are corresponding sides and $\overline{CD} \cong \overline{GH}$. Since vertex D corresponds to vertex H, and vertex A corresponds to vertex E, sides $\overline{DA}$ and $\overline{HE}$ are corresponding sides and $\overline{DA} \cong \overline{HE}$.

Hence, in these congruent polygons, *all the pairs of corresponding sides are congruent.*

Now we are ready to give a formal definition of congruent polygons which will be useful in our work in geometry.

Definition: Two polygons are congruent **if there is a one-to-one correspondence between their vertices such that:**

1. **All pairs of corresponding angles are congruent.**
2. **All pairs of corresponding sides are congruent.**

The pairs of congruent angles and the pairs of congruent sides are called *corresponding parts of the congruent polygons.* Since a definition is reversible, we can say that:

Corresponding parts of congruent polygons are congruent.

Congruent Triangles

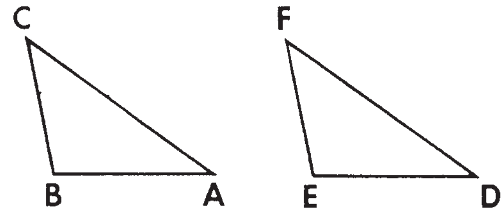


Fig. 3–3

Consider $\triangle ABC$ and $\triangle DEF$ shown in Fig. 3–3. If we match vertex A with vertex D, then A and D are a pair of *corresponding vertices*. If we match vertex B with vertex E, then B and E are a pair of corresponding vertices. Also, C and F are a pair of corresponding vertices.

In $\triangle ABC$ and $\triangle DEF$, angle A and angle D, which have corresponding vertices, are *corresponding angles*. Also, $\angle B$ and $\angle E$, as well as $\angle C$ and $\angle F$, are corresponding angles.

In $\triangle ABC$ and $\triangle DEF$, sides $\overline{AB}$ and $\overline{DE}$, which join corresponding vertices, are *corresponding sides*. Also, $\overline{BC}$ and $\overline{EF}$, as well as $\overline{CA}$ and $\overline{FD}$, are corresponding sides.

If we know that:

1. $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$
2. $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$

we can say that triangle ABC is congruent to triangle DEF, symbolized $\triangle ABC \cong \triangle DEF$.

Whenever two triangles are congruent, their corresponding sides are congruent and their corresponding angles are congruent. We can also say that when two triangles are congruent, their corresponding parts are congruent. Hence, if we know that $\triangle ABC \cong \triangle DEF$, we immediately know six different facts, which are listed below. In the column on the left, these six facts are stated as congruences; in the column on the right, the same six facts are stated as equalities.

Congruences	*Equalities*
$\overline{AB} \cong \overline{DE}$	$AB = DE$
$\overline{BC} \cong \overline{EF}$	$BC = EF$
$\overline{AC} \cong \overline{DF}$	$AC = DF$
$\angle A \cong \angle D$	$m\angle A = m\angle D$
$\angle B \cong \angle E$	$m\angle B = m\angle E$
$\angle C \cong \angle F$	$m\angle C = m\angle F$

Each congruence on the left is equivalent to the equality on its right. Hence, at any time we may use these notations interchangeably. We will use that notation which serves our purposes best in a particular situation.

For example, in one situation we may prefer to write $\overline{AC} \cong \overline{DF}$. In another situation, however, we may prefer to write $AC = DF$. Likewise, at one time we may prefer to write $\angle C \cong \angle F$; at another time we may prefer to write $m\angle C = m\angle F$.

In two congruent triangles, a pair of corresponding sides is always found opposite a pair of corresponding congruent angles. For example, in $\triangle ABC$ and $\triangle DEF$ (Fig. 3–3), corresponding sides $\overline{AB}$ and $\overline{DE}$ are opposite the pair of corresponding congruent angles, $\angle C$ and $\angle F$. Also, a pair of corresponding angles is always found opposite a pair of corresponding congruent sides. For example, corresponding angles A and D are found opposite the pair of corresponding congruent sides $\overline{BC}$ and $\overline{EF}$.

Properties of Congruence

Since congruence leads to a set of equations involving the equality of measures of angles and the equality of measures of line segments, the following properties hold for congruence as they hold for equality:

Reflexive Property for Congruence

Postulate 25. **Any geometric figure is congruent to itself.**

For example, $\triangle ABC \cong \triangle ABC$.

Symmetric Property for Congruence

Postulate 26. **A congruence may be reversed.**

For example, if $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$.

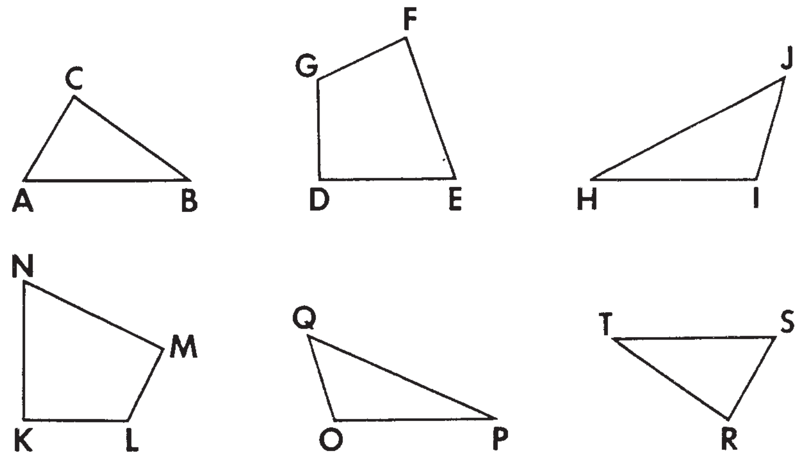
Transitive Postulate for Congruence

Postulate 27. **Two geometric figures congruent to the same geometric figure are congruent to each other.**

For example, if $\triangle ABC \cong \triangle DEF$, and $\triangle DEF \cong \triangle RST$, then it follows that $\triangle ABC \cong \triangle RST$.

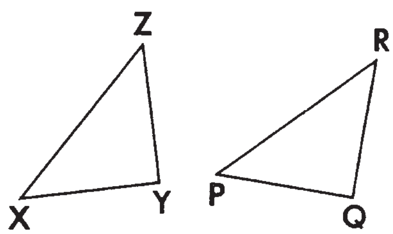
EXERCISES

1. Referring to the figures at the top of page 108: (*a*) Name the figures which appear to be congruent. (*b*) Use the symbol $\cong$ to write that the figures named are congruent.

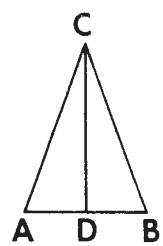


Ex. 1

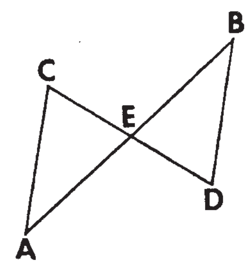
In exercises 2–5, name three pairs of corresponding angles and three pairs of corresponding sides in the given congruent triangles. In each exercise, use the symbol ≅ to indicate that the angles named and also the sides named in your answers are congruent.



Ex. 2

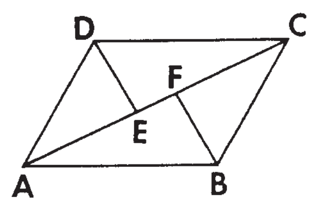


Ex. 3



Ex. 4

2. $\triangle XYZ \cong \triangle PQR$
3. $\triangle ADC \cong \triangle BDC$
4. $\triangle AEC \cong \triangle BED$
5. $\triangle CDE \cong \triangle ABF$



Ex. 5

2. Proving Triangles Congruent When They Agree in Two Sides and the Included Angle

We have seen that two triangles can be proved congruent by proving that their three pairs of corresponding angles and three pairs of corresponding sides are congruent. Now let us see whether it is possible to prove two triangles congruent by proving that fewer than three pairs of sides and three pairs of angles are congruent.

Let us perform the following experiment:

In $\triangle ABC$ (see Fig. 3–4), we see that $AB = 1$ inch, $m\angle A = 45$, and $AC = \frac{3}{4}$ inch. We say that $\angle A$ is *included* between side $\overline{AB}$ and side $\overline{AC}$, because these two segments are on the sides of the angle.

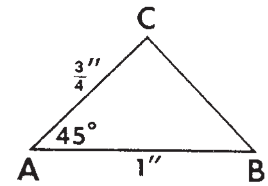


Fig. 3–4

On a sheet of paper, let us draw $\triangle A'B'C'$ so that $A'B' = 1$ inch, $A'C' = \frac{3}{4}$ inch, and the measure of the included angle A' is 45, $m\angle A' = 45$. (See Fig. 3–5.)

(1) We begin by drawing a working line on which we measure off 1 inch, the length of $\overline{A'B'}$. (2) With a protractor, we draw an angle of 45° whose vertex is at point A'. (3) On the side of $\angle A'$ which was last drawn, we measure off a line segment $\frac{3}{4}$ of an inch in length, beginning at point A' and ending at point C'. (4) We then draw side $\overline{C'B'}$ to complete the triangle.

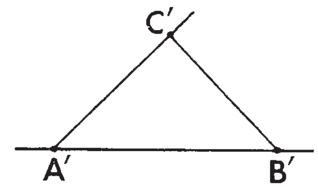


Fig. 3–5

If we measure sides $\overline{CB}$ and $\overline{C'B'}$, we find the measures equal. Hence, $\overline{CB} \cong \overline{C'B'}$. Also, if we measure $\angle C$ and $\angle C'$, we find their measures equal. $\angle B$ and $\angle B'$, if measured, are also found to have equal measures. Hence, $\angle C \cong \angle C'$ and $\angle B \cong \angle B'$. Also, if we cut out $\triangle A'B'C'$, we can make it coincide with $\triangle ABC$. Thus, $\triangle A'B'C'$ appears to be congruent to $\triangle ABC$.

If we repeat the same experiment several times with different sets of measurements for the two sides and the included angle, in each experiment the remaining pairs of corresponding parts of the triangles will appear to be congruent, and the triangles themselves will appear to be congruent. It seems reasonable, therefore, to accept the following statement, whose truth we will assume without proof:

Postulate 28. **Two triangles are congruent if two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of the other. [s.a.s. $\cong$ s.a.s.]**

In $\triangle ABC$ and $\triangle A'B'C'$ (Fig. 3–6), if $\overline{AB} \cong \overline{A'B'}$, $\angle A \cong \angle A'$, $\overline{AC} \cong \overline{A'C'}$, then $\triangle ABC \cong \triangle A'B'C'$.

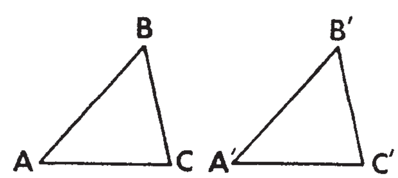


Fig. 3–6

NOTE. In the statement of postulate 28, we did not take the trouble to state that there must exist a correspondence between the vertices of the two triangles such that two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of the other. In the future, we will follow the same practice when we state other postulates or theorems involving congruent triangles. In each case, we will understand that there does exist a correspondence between the vertices of the two triangles for which the congruences stated in the hypothesis exist.

Definition. **A *corollary* is a theorem that can easily be deduced from another theorem or from a postulate.**

The following statement is a corollary of the preceding postulate:

Corollary P28–1. **Two right triangles are congruent if the legs of one right triangle are congruent to the legs of the other right triangle.**

In right triangles ABC and $A'B'C'$ (Fig. 3–7) with right angles at C and C', if leg $\overline{BC} \cong$ leg $\overline{B'C'}$ and leg $\overline{AC} \cong$ leg $\overline{A'C'}$, then $\triangle ABC \cong \triangle A'B'C'$.

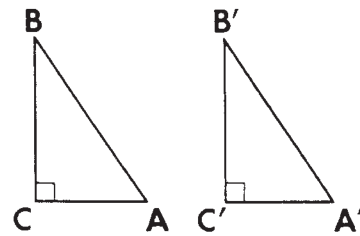


Fig. 3–7

Now we will see how the preceding postulate can be used to prove two triangles congruent. When we are proving triangles congruent, we can use the following device to help us show the method of congruence being used. We will write (s. ≅ s.) next to pairs of congruent sides and (a. ≅ a.) next to pairs of congruent angles that are used to establish the congruence of the triangles.

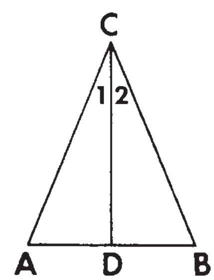
MODEL PROBLEM

In $\triangle ABC$, if $\overline{AC} \cong \overline{BC}$ and $\overline{CD}$ bisects $\angle ACB$, prove that $\triangle ACD \cong \triangle BCD$.

Given: $\triangle ABC$ with $\overline{AC} \cong \overline{BC}$.
$\overline{CD}$ bisects $\angle ACB$.

To prove: $\triangle ACD \cong \triangle BCD$.

Plan: Prove the triangles congruent by showing that s.a.s. ≅ s.a.s.

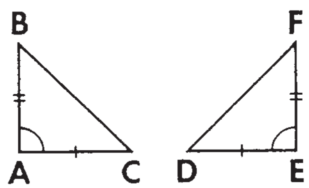


Proof:

Statements	*Reasons*
1. In $\triangle ABC$, $\overline{AC} \cong \overline{BC}$. (s. ≅ s.)	1. Given.
2. $\overline{CD}$ bisects $\angle ACB$.	2. Given.
3. $\angle 1 \cong \angle 2$. (a. ≅ a.)	3. An angle bisector divides the angle into two congruent angles.
4. $\overline{CD} \cong \overline{CD}$. (s. ≅ s.)	4. Reflexive property of congruence.
5. $\triangle ACD \cong \triangle BCD$.	5. s.a.s. ≅ s.a.s.

EXERCISES

In 1–6, tell whether or not the triangles can be proved congruent and give the reason for your answer. (Pairs of line segments marked with the same number of strokes are congruent. Pairs of angles marked with the same number of arcs are congruent. A line segment or an angle marked with × is congruent to itself by the reflexive property of congruence.)



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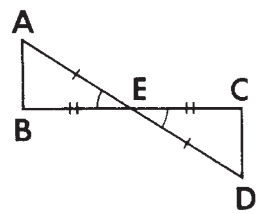

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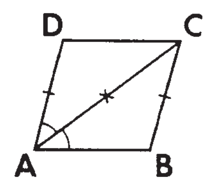

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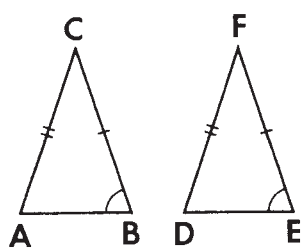

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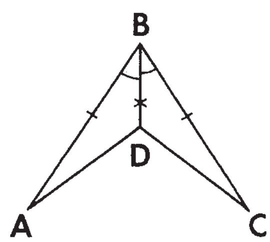

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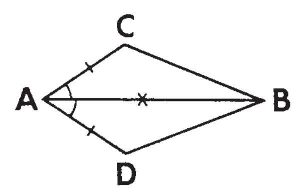
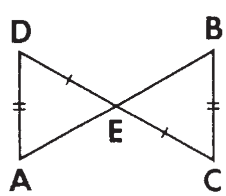


6.

In 7–9, name the pair of corresponding sides or the pair of corresponding angles that would have to be proved congruent in addition to those pairs marked congruent in order to prove that the triangles are congruent by s.a.s. ≅ s.a.s.



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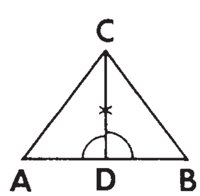

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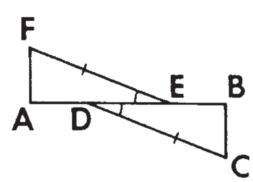
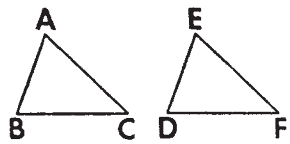


9.

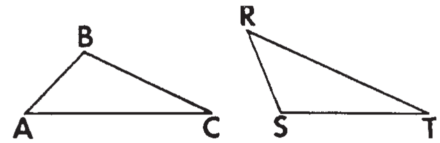
Exercises 10 and 11 refer to the figures on the next page.

10. *Given:* $AB = 4$, $ED = 4$, $BC = 6$, $DF = 6$, $m\angle B = 70$, $m\angle D = 70$.
Prove: $\triangle ABC \cong \triangle EDF$.

11. *Given:* $AB = 12$, $RS = 12$, $m\angle B = 120$, $m\angle S = 120$, $\overline{BC} \cong \overline{ST}$.
Prove: $\triangle ABC \cong \triangle RST$.

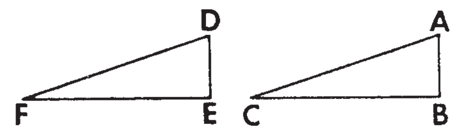


Ex. 10



Ex. 11

12. *Given:* $\overline{DE} \cong \overline{AB}$, $\overline{EF} \cong \overline{BC}$, $\angle E$ and $\angle B$ are right angles.
Prove: $\triangle DEF \cong \triangle ABC$.



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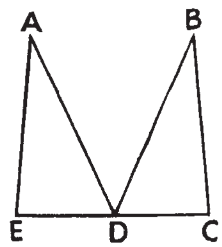

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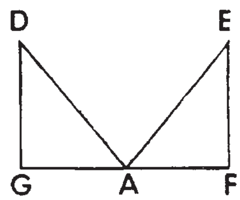

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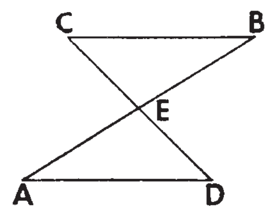
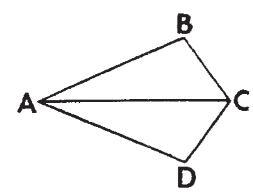
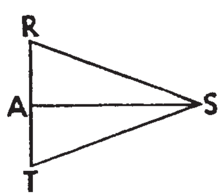


Ex. 15

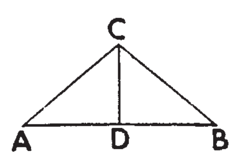


Ex. 16

13. *Given:* $\overline{AE} \cong \overline{BC}$, $\angle E \cong \angle C$, D is the midpoint of $\overline{EC}$.
Prove: $\triangle ADE \cong \triangle BDC$.
14. *Given:* $\overline{DG} \cong \overline{EF}$, A is the midpoint of $\overline{GF}$, $\overline{DG} \perp \overline{GF}$, $\overline{EF} \perp \overline{GF}$.
Prove: $\triangle DGA \cong \triangle EFA$.
15. *Given:* $\overline{CD}$ bisects $\overline{AB}$, $\overline{AB}$ bisects $\overline{CD}$.
Prove: $\triangle AED \cong \triangle BEC$.
16. *Given:* $\overline{AB} \cong \overline{AD}$, $\overline{AC}$ bisects $\angle BAD$.
Prove: $\triangle ABC \cong \triangle ADC$.



Ex. 17



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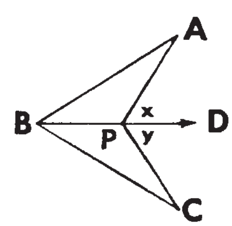

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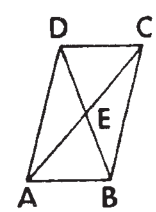


Ex. 20

17. *Given:* $\overline{AS} \perp \overline{RT}$, A is the midpoint of $\overline{RT}$.
Prove: $\triangle RAS \cong \triangle TAS$.
18. *Given:* $\overline{CD}$ is the $\perp$ bisector of $\overline{AB}$.
Prove: $\triangle ADC \cong \triangle BDC$.
19. If $\overline{AP} \cong \overline{CP}$, $\angle x \cong \angle y$, and $\overleftrightarrow{BD}$ is a straight line, prove that $\triangle ABP \cong \triangle CBP$.
20. If $\overline{DB}$ and $\overline{AC}$ bisect each other, prove that $\triangle AEB \cong \triangle CED$.

3. Proving Triangles Congruent When They Agree in Two Angles and the Included Side

Let us perform the following experiment:

In $\triangle ABC$ (see Fig. 3–8), we see that $m\angle A = 60$, $AB = 1$ inch, and $m\angle B = 50$. We say that side $\overline{AB}$ is *included* between $\angle A$ and $\angle B$, because side $\overline{AB}$ has vertex A and vertex B as its endpoints.

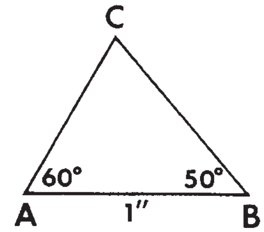


Fig. 3–8

On a sheet of paper, let us draw $\triangle A'B'C'$ so that $m\angle A' = 60$, $m\angle B' = 50$, and the length of the included side $\overline{A'B'}$ is 1 inch, $A'B' = 1$ inch. (See Fig. 3–9.)

(1) We begin by drawing a working line on which we measure off 1 inch, the length of $\overline{A'B'}$. (2) With a protractor, we draw an angle of 60° whose vertex is at point A'. (3) We then draw an angle of 50° whose vertex is at point B'. (4) To complete the triangle, we extend the sides of these angles until they intersect at point C'.

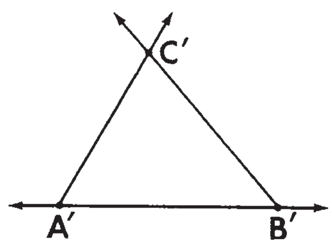


Fig. 3–9

If we measure sides $\overline{AC}$ and $\overline{A'C'}$, we find their lengths to be equal. Also, if we measure $\overline{CB}$ and $\overline{C'B'}$, we find their lengths to be equal. Hence, $\overline{AC} \cong \overline{A'C'}$ and $\overline{CB} \cong \overline{C'B'}$. $\angle C$ and $\angle C'$, if measured, are also found to have equal measures. Hence, $\angle C \cong \angle C'$. Furthermore, if we cut out $\triangle A'B'C'$, we can make it coincide with $\triangle ABC$. Therefore, it appears that $\triangle A'B'C'$ is congruent to $\triangle ABC$.

If we repeat the same experiment several times with different sets of measurements for the two angles and the included side, in each experiment the remaining pairs of corresponding parts of the triangles will appear to be congruent and the triangles themselves will appear to be congruent. It seems reasonable, therefore, to accept the following statement, whose truth we will assume without proof:

Postulate 29. **Two triangles are congruent if two angles and the included side of one triangle are congruent respectively to two angles and the included side of the other. [a.s.a. $\cong$ a.s.a.]**

If $\angle A \cong \angle A'$, $\overline{AC} \cong \overline{A'C'}$, $\angle C \cong \angle C'$, then $\triangle ABC \cong \triangle A'B'C'$. (See Fig. 3–10.)

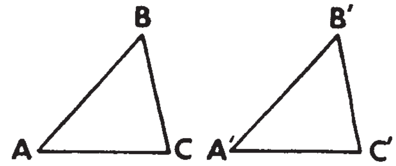


Fig. 3–10

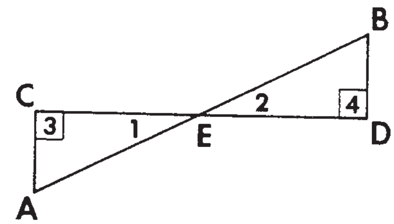
Now we will see how the preceding postulate can be used to prove two triangles congruent.

MODEL PROBLEM

Given: $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ are straight lines which intersect at E. $\overline{BA}$ bisects $\overline{CD}$. $\overline{AC} \perp \overline{CD}$, $\overline{BD} \perp \overline{CD}$

To prove: $\triangle ACE \cong \triangle BDE$.

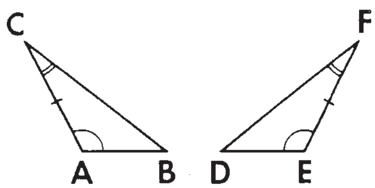
Plan: Prove the triangles congruent by showing that a.s.a. ≅ a.s.a.



Proof: *Statements*	*Reasons*
1. $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ are straight lines which intersect at E.	1. Given.
2. $\angle 1$ and $\angle 2$ are vertical angles.	2. Definition of vertical angles.
3. $\angle 1 \cong \angle 2$. (a. ≅ a.)	3. If two angles are vertical angles, they are congruent.
4. $\overline{BA}$ bisects $\overline{CD}$.	4. Given.
5. $\overline{CE} \cong \overline{DE}$. (s. ≅ s.)	5. A bisector divides a line segment into two congruent parts.
6. $\overline{AC} \perp \overline{CD}$, $\overline{BD} \perp \overline{CD}$.	6. Given.
7. $\angle 3$ and $\angle 4$ are right angles.	7. Perpendicular lines are lines which intersect and form right angles.
8. $\angle 3 \cong \angle 4$. (a. ≅ a.)	8. If two angles are right angles, they are congruent.
9. $\triangle ACE \cong \triangle BDE$.	9. a.s.a. ≅ a.s.a.

EXERCISES

In 1–6, tell whether or not the triangles can be proved congruent by the a.s.a. ≅ a.s.a. postulate, using only the marked congruent parts in establishing the congruence. Give the reason for your answer.



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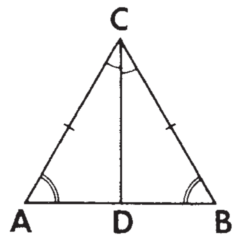

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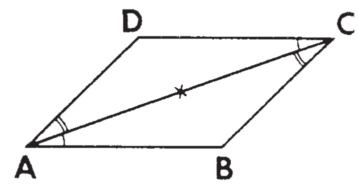

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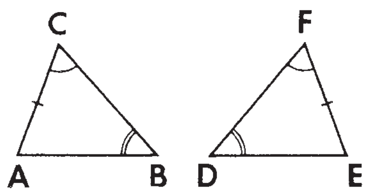

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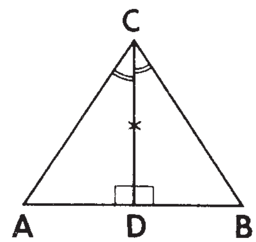

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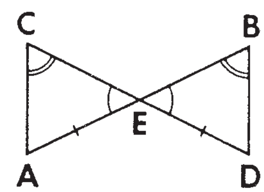
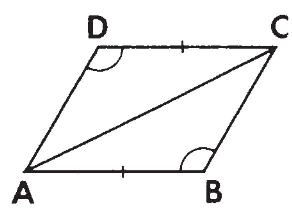


6.

In 7–9, name the pair of corresponding sides or the pair of corresponding angles that would have to be proved congruent in addition to those pairs marked congruent in order to prove that the triangles are congruent by a.s.a. $\cong$ a.s.a.



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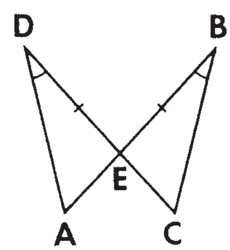

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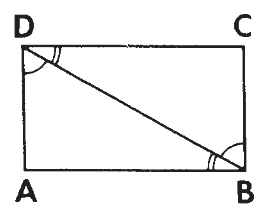

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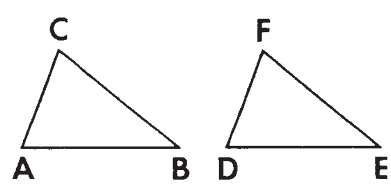
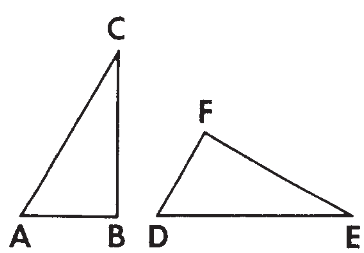
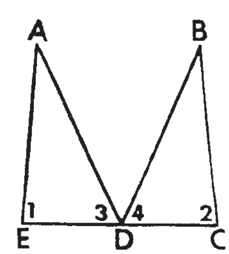


Ex. 10



Ex. 11

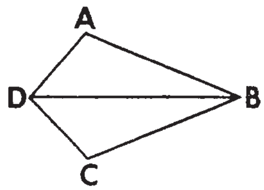


Ex. 12

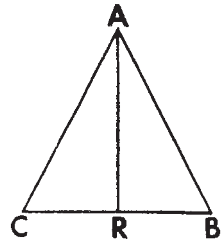
10. *Given:* $AB = 4$, $DE = 4$, $m\angle A = 70$, $m\angle D = 70$, $m\angle B = 50$, $m\angle E = 50$.
Prove: $\triangle ABC \cong \triangle DEF$.

11. *Given:* $AB = 8$, $DF = 8$, $m\angle B = 90$, $m\angle F = 90$, $m\angle A = 60$, $m\angle D = 60$.
Prove: $\triangle ABC \cong \triangle DFE$.

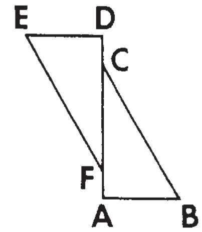
12. *Given:* $\angle 1 \cong \angle 2$, D is the midpoint of $\overline{EC}$, $\angle 3 \cong \angle 4$.
Prove: $\triangle AED \cong \triangle BCD$.



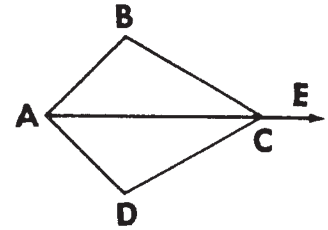
Ex. 13



Ex. 14



Ex. 15



Ex. 16

13. *Given:* $\overline{DB}$ bisects $\angle ADC$, $\overline{BD}$ bisects $\angle ABC$.
Prove: $\triangle ADB \cong \triangle CDB$.

14. *Given:* $\overline{AR} \perp \overline{CB}$, $\overline{AR}$ bisects $\angle CAB$.
Prove: $\triangle ACR \cong \triangle ABR$.

15. *Given:* $\overleftrightarrow{DA}$ is a straight line, $\angle E \cong \angle B$, $\overline{ED} \cong \overline{AB}$, $\overline{FD} \perp \overline{DE}$, $\overline{CA} \perp \overline{AB}$.
Prove: $\triangle DEF \cong \triangle ABC$.

16. *Given:* $\overrightarrow{AE}$ bisects $\angle BAD$, $\angle ECB \cong \angle ECD$.
Prove: $\triangle ABC \cong \triangle ADC$.

4. Proving Triangles Congruent When They Agree in Three Sides

Let us perform the following experiment:

In $\triangle ABC$ (see Fig. 3–11), we see that $AB = 1$ inch, $BC = \frac{7}{8}$ inch, and $AC = \frac{3}{4}$ inch.

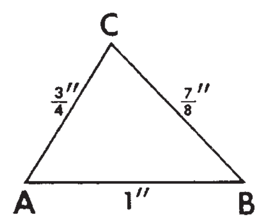


Fig. 3–11

On a sheet of paper, let us draw $\triangle A'B'C'$ so that the length of $\overline{A'B'}$ is 1 inch ($A'B' = 1$ inch), the length of $\overline{B'C'}$ is $\frac{7}{8}$ inch ($B'C' = \frac{7}{8}$ inch), and the length of $\overline{A'C'}$ is $\frac{3}{4}$ inch ($A'C' = \frac{3}{4}$ inch). (See Fig. 3–12.)

(1) We begin by drawing a working line on which we measure off 1 inch, the length of $\overline{A'B'}$. (2) Using point A' as a center, we draw an arc of a circle whose radius is $\frac{3}{4}$ of an inch in length. (3) Using point B' as a center, we draw an arc of a circle whose radius is $\frac{7}{8}$ of an inch in length. This arc intersects the first arc at point C'. (4) We now draw sides $\overline{A'C'}$ and $\overline{B'C'}$ to complete the triangle.

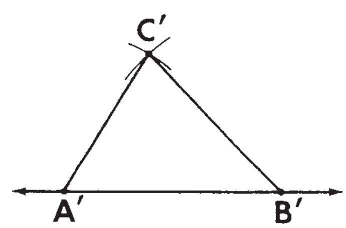


Fig. 3–12

If we measure $\angle A$ and $\angle A'$, we find their measures equal. Hence, $\angle A \cong \angle A'$. If we measure $\angle B$ and $\angle B'$, we find their measures equal. Hence, $\angle B \cong \angle B'$. If we measure $\angle C$ and $\angle C'$, we find their measures equal. Hence, $\angle C \cong \angle C'$. Also, if we cut out $\triangle A'B'C'$, we can make it coincide with $\triangle ABC$. Thus, $\triangle A'B'C'$ appears to be congruent to $\triangle ABC$.

If we repeat the same experiment several times with different sets of measurements for the three sides, in each experiment the triangles will appear to be congruent. It seems reasonable, therefore, to accept the following statement, whose truth we will assume without proof:

Postulate 30. **Two triangles are congruent if the three sides of one triangle are congruent respectively to the three sides of the other. [s.s.s. ≅ s.s.s.]**

If $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, $\overline{BC} \cong \overline{B'C'}$, then $\triangle ABC \cong \triangle A'B'C'$. (See Fig. 3–13.)

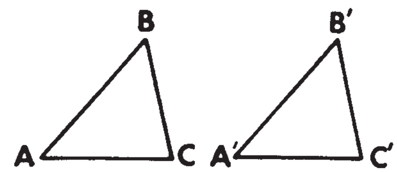


Fig. 3–13

NOTE. After theorem 9 (page 747) and theorem 10 (pages 748–749) have been proved, it is possible to prove "Two triangles are congruent if the three sides of one triangle are congruent respectively to the three sides of the other." However, since the proof is detailed, long, and difficult for the student at this early stage of the geometry course, we prefer to postulate the s.s.s. ≅ s.s.s. congruence.

Now we will see how the preceding postulate can be used to prove two triangles congruent.

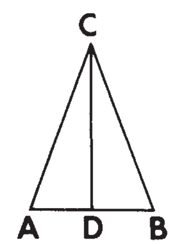
MODEL PROBLEM

If a median is drawn to the base of an isosceles triangle, prove that the median divides the triangle into two congruent triangles.

Given: Isosceles triangle ABC with $\overline{CA} \cong \overline{CB}$. $\overline{CD}$ is a median to base $\overline{AB}$.

To prove: $\triangle ACD \cong \triangle BCD$.

Plan: Prove the triangles congruent by showing that s.s.s. ≅ s.s.s.

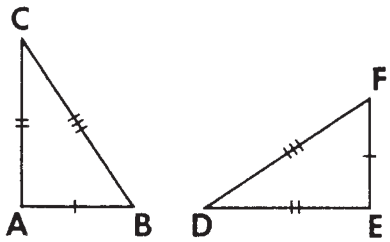


Proof:

Statements	*Reasons*
1. In isosceles triangle ABC, $\overline{CA} \cong \overline{CB}$. (s. ≅ s.)	1. Given.
2. $\overline{CD}$ is the median to base $\overline{AB}$.	2. Given.
3. $\overline{AD} \cong \overline{BD}$. (s. ≅ s.)	3. A median in a triangle divides the side to which it is drawn into two congruent parts.
4. $\overline{CD} \cong \overline{CD}$. (s. ≅ s.)	4. Reflexive property of congruence.
5. $\triangle ACD \cong \triangle BCD$.	5. s.s.s. ≅ s.s.s.

EXERCISES

In 1–6, tell whether or not the triangles can be proved congruent using only the marked congruent parts in establishing the congruence. Give the reason for your answer.



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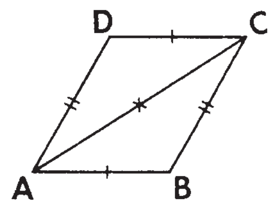

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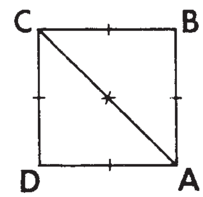

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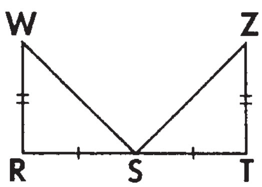

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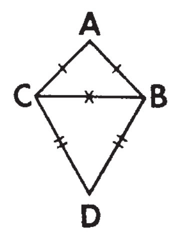

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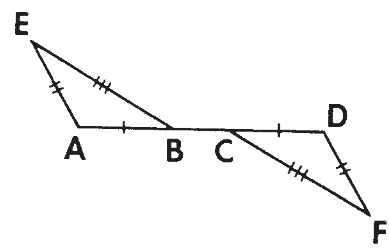
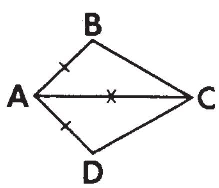


6.

In 7–9, name the pair of corresponding sides that would have to be proved congruent in addition to those pairs marked congruent in order to prove that the triangles are congruent by s.s.s. ≅ s.s.s.



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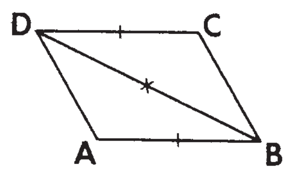

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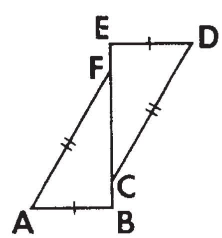
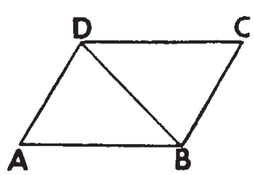
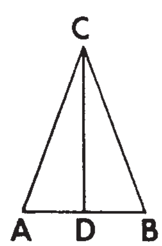


9.

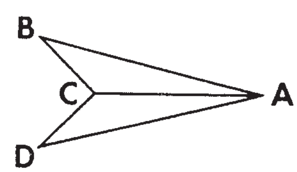
Exercises 10–12 on the next page refer to the following figures:



Ex. 10



Ex. 11

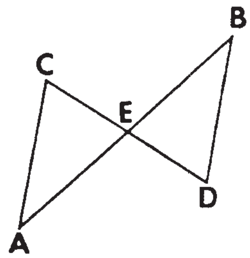


Ex. 12

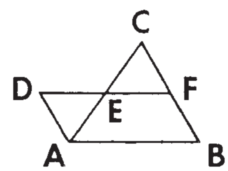
10. *Given:* $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{CB}$.
Prove: $\triangle ABD \cong \triangle CDB$.

11. *Given:* In $\triangle ABC$, $\overline{CA} \cong \overline{CB}$, D is the midpoint of $\overline{AB}$.
Prove: $\triangle ADC \cong \triangle BDC$.

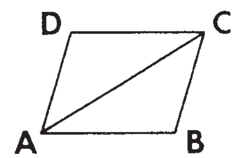
12. *Given:* $\overline{AB} \cong \overline{AD}$, $\overline{CB} \cong \overline{CD}$.
Prove: $\triangle ABC \cong \triangle ADC$.



Ex. 13



Ex. 14



Ex. 15

13. *Given:* E is the midpoint of $\overline{CD}$, $\overline{CD}$ bisects $\overline{AB}$, $\overline{AC} \cong \overline{BD}$.
Prove: $\triangle AEC \cong \triangle BED$.

14. *Given:* $\overline{AC}$ and $\overline{DF}$ bisect each other at E, $\overline{AD} \cong \overline{CF}$.
Prove: $\triangle DEA \cong \triangle FEC$.

15. If both pairs of opposite sides of quadrilateral $ABCD$ (sides that do not have a common endpoint) are congruent, prove that $\triangle ABC \cong \triangle CDA$.

5. More Practice in Proving Triangles Congruent

Methods of Proving Triangles Congruent

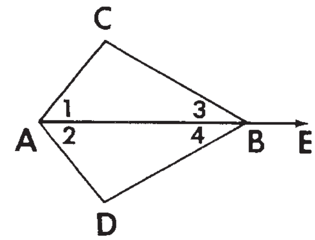
To prove that two triangles are congruent, prove that any one of the following statements is true:

1. Two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of the other. [s.a.s. ≅ s.a.s.]
2. Two angles and the included side of one triangle are congruent respectively to two angles and the included side of the other. [a.s.a. ≅ a.s.a.]
3. Three sides in one triangle are congruent respectively to the three sides of the other. [s.s.s. ≅ s.s.s.]

Analyzing a Congruence Problem

A process of *analysis* can help us to determine which of the three postulates can be used to prove that two triangles are congruent. Let us see how to perform such an analysis for the following congruence problems:

MODEL PROBLEMS



1. *Given:* $\overleftrightarrow{AE}$ is a straight line which bisects $\angle CAD$. $\angle CBE \cong \angle DBE$.

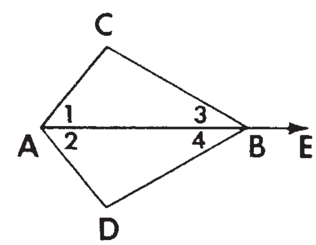
To prove: $\triangle ACB \cong \triangle ADB$.

Since $\overleftrightarrow{AE}$ bisects $\angle CAD$, $\angle 1 \cong \angle 2$, giving us one pair of congruent angles. We see that $\overline{AB}$ is a common side in both triangles. Therefore, we can say that $\overline{AB} \cong \overline{AB}$, by the reflexive property of congruence, giving us a pair of congruent sides. Since we have proved one pair of sides congruent and one pair of angles congruent, we may be able to use either the s.a.s. postulate or the a.s.a. postulate to establish the congruence.

In order to use the s.a.s. postulate, we would have to prove that $\overline{AC} \cong \overline{AD}$. Since we have no information about these sides, it appears that we will probably not be able to use the s.a.s. postulate.

In order to use the a.s.a. postulate, we would have to prove that $\angle 3 \cong \angle 4$. Since we know from the *given* that $\angle CBE \cong \angle DBE$ and that $\overleftrightarrow{AE}$ is a straight line, we can show that $\angle 3$, which is the supplement of $\angle CBE$, must be congruent to $\angle 4$, which is the supplement of $\angle DBE$, because "two angles that are supplementary to congruent angles are congruent."

Therefore, we see that we can use the a.s.a. postulate to prove $\triangle ACB \cong \triangle ADB$. The formal proof follows:



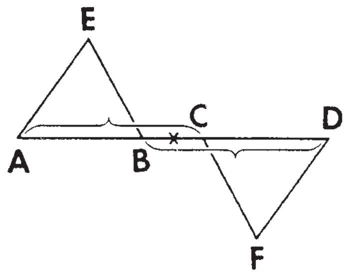
Given: $\overleftrightarrow{AE}$ bisects $\angle CAD$.
$\angle CBE \cong \angle DBE$.

To prove: $\triangle ACB \cong \triangle ADB$.

Proof:

Statements	*Reasons*
1. $\overleftrightarrow{AE}$ bisects $\angle CAD$.	1. Given.
2. $\angle 1 \cong \angle 2$. (a. $\cong$ a.)	2. An angle bisector divides the angle into two congruent angles.
3. $\overline{AB} \cong \overline{AB}$ (s. $\cong$ s.)	3. Reflexive property of congruence.
4. $\overleftrightarrow{AE}$ is a straight line.	4. Given.
5. $\angle CBE \cong \angle DBE$.	5. Given.
6. $\angle 3$ is supplementary to $\angle CBE$. $\angle 4$ is supplementary to $\angle DBE$.	6. If the non-common sides of two adjacent angles lie on a straight line, the angles are supplementary.
7. $\angle 3 \cong \angle 4$. (a. $\cong$ a.)	7. If two angles are supplements of congruent angles, they are congruent.
8. $\triangle ACB \cong \triangle ADB$.	8. a.s.a. $\cong$ a.s.a.

2. *Given:* $\overleftrightarrow{AD}$ is a straight line.
$\overline{AE} \cong \overline{DF}$.
$\angle A \cong \angle D$.
$\overline{AC} \cong \overline{DB}$.



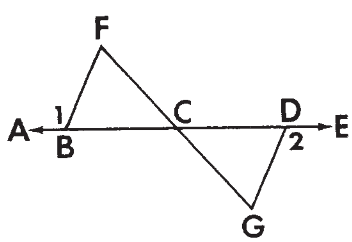
To prove: $\triangle AEB \cong \triangle DFC$.

Plan: Prove the triangles congruent by showing that s.a.s. $\cong$ s.a.s. In order to do this, it is necessary to prove $\overline{AB} = \overline{DC}$.

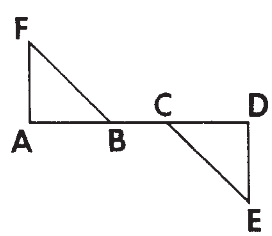
Proof:

Statements	*Reasons*
1. $\overline{AE} \cong \overline{DF}$. (s. $\cong$ s.)	1. Given.
2. $\angle A \cong \angle D$. (a. $\cong$ a.)	2. Given.
3. $\overleftrightarrow{AD}$ is a straight line.	3. Given.
4. $\overline{AC} \cong \overline{DB}$.	4. Given.
5. $\overline{BC} \cong \overline{BC}$.	5. Reflexive property of congruence.
6. $\overline{AC} - \overline{BC} \cong \overline{DB} - \overline{BC}$, or $\overline{AB} \cong \overline{DC}$. (s. $\cong$ s.)	6. If congruent segments are subtracted from congruent segments, the differences are congruent segments.
7. $\triangle AEB \cong \triangle DFC$.	7. s.a.s. $\cong$ s.a.s.

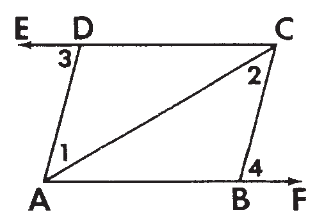
EXERCISES



Ex. 1



Ex. 2



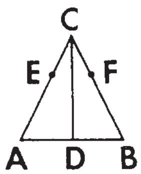
Ex. 3

1. *Given:* $\overleftrightarrow{AE}$ and $\overleftrightarrow{FG}$ are straight lines, C is the midpoint of $\overline{BD}$, $\angle 1 \cong \angle 2$.
Prove: $\triangle BFC \cong \triangle DGC$.
2. *Given:* $\overleftrightarrow{AD}$ is a straight line, $\overline{FA} \perp \overline{AD}$, $\overline{ED} \perp \overline{AD}$, $\overline{AF} \cong \overline{DE}$, $\overline{AC} \cong \overline{DB}$.
Prove: $\triangle ABF \cong \triangle DCE$.
3. *Given:* $\overleftrightarrow{EC}$ and $\overleftrightarrow{AF}$ are straight lines, $\overline{AD} \cong \overline{CB}$, $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$.
Prove: $\triangle ADC \cong \triangle CBA$.

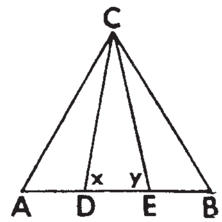
4. *Prove:* Two right triangles are congruent if the legs of one triangle are congruent to the legs of the other triangle.

5. *Given:* In triangle ABC, $\overline{CD}$ is the median to $\overline{AB}$, $\overline{CE} \cong \overline{CF}$, $\overline{EA} \cong \overline{FB}$.
Prove: $\triangle ACD \cong \triangle BCD$.

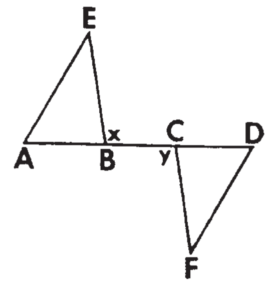
6. *Given:* In the figure, points D and E divide segment $\overline{AB}$ into three congruent parts, $\overline{CD} \cong \overline{CE}$, $\angle x \cong \angle y$.
Prove: $\triangle ACD \cong \triangle BCE$.



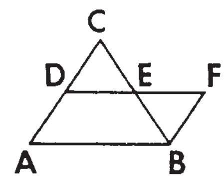
Ex. 5



Ex. 6



Ex. 8

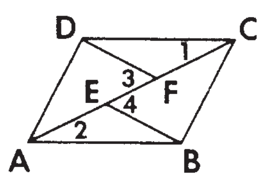


Ex. 9

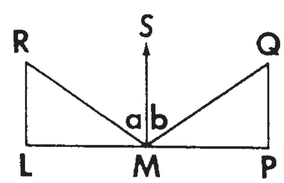
7. *Prove:* A diagonal of an equilateral quadrilateral (a quadrilateral all of whose sides are congruent) divides the quadrilateral into two congruent triangles.

8. In the figure, if $\overleftrightarrow{AD}$ is a straight line, $\overline{AC} \cong \overline{DB}$, $\angle A \cong \angle D$, and $\angle x \cong \angle y$, prove that $\triangle AEB \cong \triangle DFC$.

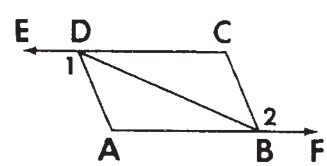
9. *Given:* In the figure, E is the midpoint of $\overline{BC}$, $\angle ACB \cong \angle FBC$, $\overline{AD} \cong \overline{CD}$, $\overline{FB} \cong \overline{AD}$.
Prove: $\triangle CDE \cong \triangle BFE$.



Ex. 10



Ex. 11



Ex. 12

10. *Given:* $\overleftrightarrow{AC}$, $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$, $\overline{AF} \cong \overline{CE}$.
Prove: $\triangle ABE \cong \triangle CDF$.

11. *Given:* $\overleftrightarrow{SM}$ is the perpendicular bisector of $\overline{LP}$, $\overline{RM} \cong \overline{QM}$, $\angle a \cong \angle b$.
Prove: $\triangle RLM \cong \triangle QPM$.

12. *Given:* $\overleftrightarrow{EC}$ and $\overleftrightarrow{AF}$ are straight lines, $\angle 1 \cong \angle 2$, $\angle EDB \cong \angle FBD$.
Prove: $\triangle ADB \cong \triangle CBD$.

6. Proving Overlapping Triangles Congruent

If we are given that $\overline{AD} \cong \overline{BC}$ and $\overline{DB} \cong \overline{CA}$, can we prove that $\triangle DAB \cong \triangle CBA$?

Since these two triangles overlap, we may find it easier to visualize them if we use any one of the following devices:

1. Outline one of the triangles with a solid line, the other with a dotted line, as shown in Fig. 3–14.
2. Outline the triangles, using two contrasting colored crayons.
3. Separate the triangles, as shown in Fig. 3–15.

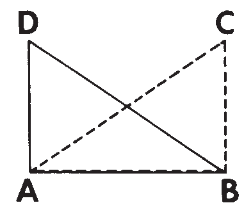


Fig. 3–14

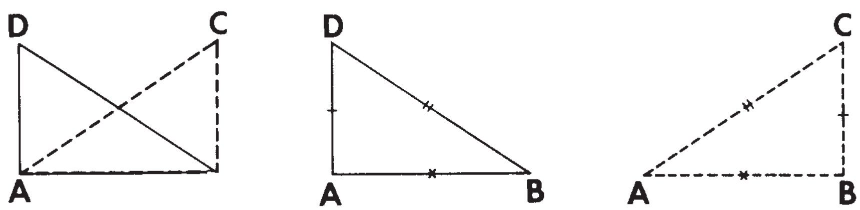


Fig. 3–15

Now we can see more clearly the relationships in $\triangle DAB$ and $\triangle CBA$. We were given that $\overline{AD} \cong \overline{BC}$ and $\overline{DB} \cong \overline{CA}$. Since $\overline{AB} \cong \overline{AB}$ by the reflexive property of congruence, $\triangle DAB \cong \triangle CBA$ by s.s.s. $\cong$ s.s.s.

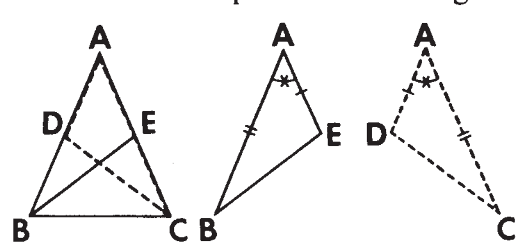
MODEL PROBLEM

Given: In $\triangle ABC$, $\overline{AB} \cong \overline{AC}$. $\overline{CD}$ and $\overline{BE}$ are medians.

To prove: $\triangle ABE \cong \triangle ACD$.

Plan: Prove the triangles congruent by showing that s.a.s. $\cong$ s.a.s.

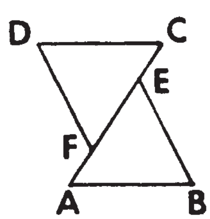
Separate the Triangles



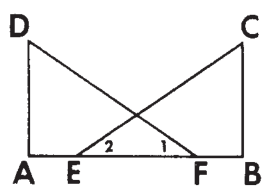
[The proof is given on the next page.]

Proof: *Statements*	*Reasons*
1. $\overline{AB} \cong \overline{AC}$. (s. ≅ s.)	1. Given.
2. $\angle A \cong \angle A$. (a. ≅ a.)	2. Reflexive property of congruence.
3. $\overline{CD}$ and $\overline{BE}$ are medians.	3. Given.
4. $AD = \frac{1}{2}AB$. $AE = \frac{1}{2}AC$.	4. A median divides the side to which it is drawn into two congruent parts.
5. $AD = AE$.	5. Halves of equal quantities are equal.
6. $\overline{AD} \cong \overline{AE}$. (s. ≅ s.)	6. Definition of congruent segments.
7. $\triangle ABE \cong \triangle ACD$.	7. s.a.s. ≅ s.a.s.

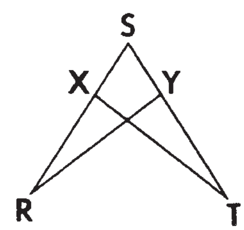
EXERCISES



Ex. 1



Ex. 2



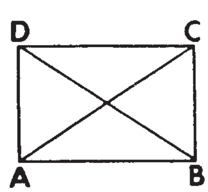
Ex. 3

1. If $\overleftrightarrow{AC}$ is a straight line, $\overline{DC} \cong \overline{BA}$, $\overline{DF} \cong \overline{BE}$, and $\overline{CE} \cong \overline{AF}$, prove that $\triangle AEB \cong \triangle CFD$.

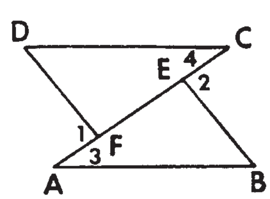
2. *Given:* $\overleftrightarrow{AB}$, $\overline{CE} \cong \overline{DF}$, $\angle 1 \cong \angle 2$, $\overline{AE} \cong \overline{BF}$.
Prove: $\triangle AFD \cong \triangle BEC$.

3. *Given:* $\overleftrightarrow{SR}$ and $\overleftrightarrow{ST}$ are straight lines, $\overline{SX} \cong \overline{SY}$, $\overline{XR} \cong \overline{YT}$.
Prove: $\triangle RSY \cong \triangle TSX$.

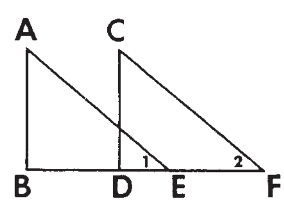
Exercises 4–6 on the next page refer to the following figures:



Ex. 4

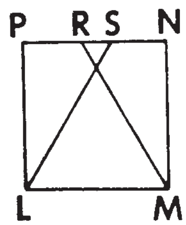


Ex. 5

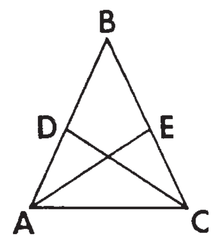


Ex. 6

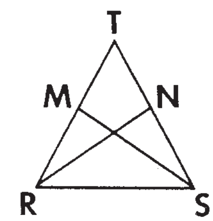
4. *Given:* $\overline{DA} \cong \overline{CB}$, $\overline{DA} \perp \overline{AB}$, $\overline{CB} \perp \overline{AB}$.
   Prove: $\triangle DAB \cong \triangle CBA$.
5. *Given:* $\overleftrightarrow{AC}$, $\overline{AF} \cong \overline{EC}$, $\angle 3 \cong \angle 4$, $\angle 1 \cong \angle 2$.
   Prove: $\triangle ABE \cong \triangle CDF$.
6. *Given:* $\overline{AB} \perp \overline{BF}$, $\overline{CD} \perp \overline{BF}$, $\overline{BD} \cong \overline{FE}$, $\angle 1 \cong \angle 2$.
   Prove: $\triangle ABE \cong \triangle CDF$.



Ex. 7

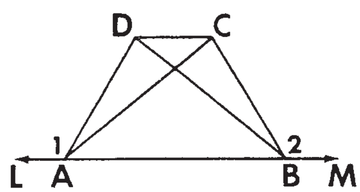


Ex. 8

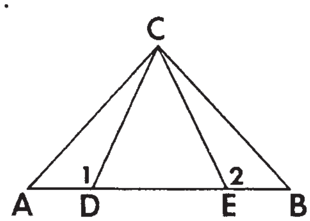


Ex. 9

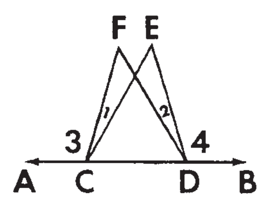
7. *Given:* $\overline{LP} \perp \overline{PN}$, $\overline{MN} \perp \overline{PN}$, $\overline{LP} \cong \overline{MN}$, $\overline{PR} \cong \overline{NS}$.
   Prove: $\triangle LPS \cong \triangle MNR$.
8. *Given:* $\angle BAC \cong \angle BCA$, $\overline{CD}$ bisects $\angle BCA$, $\overline{AE}$ bisects $\angle BAC$.
   Prove: $\triangle ADC \cong \triangle CEA$.
9. *Given:* $\overline{TR} \cong \overline{TS}$, $\overline{MR} \cong \overline{NS}$.
   Prove: $\triangle RTN \cong \triangle STM$.



Ex. 10



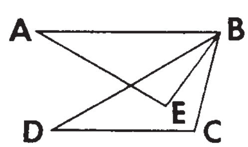
Ex. 11



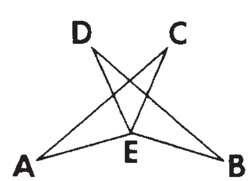
Ex. 12

10. *Given:* $\overleftrightarrow{LM}$, $\overline{CB} \cong \overline{DA}$, $\angle 2 \cong \angle 1$.
    Prove: $\triangle ABC \cong \triangle BAD$.
11. *Given:* $\overleftrightarrow{AB}$, $\overline{AD} \cong \overline{EB}$, $\angle A \cong \angle B$, $\angle 1 \cong \angle 2$.
    Prove: $\triangle AEC \cong \triangle BDC$.
12. *Given:* $\overleftrightarrow{AB}$, $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$.
    Prove: $\triangle CFD \cong \triangle DEC$.

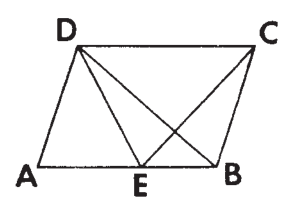
Exercises 13–15 on the next page refer to the following figures:



Ex. 13



Ex. 14



Ex. 15

13. *Given:* $\overline{AB} \cong \overline{DB}$, $\angle A \cong \angle D$, $\angle DBA \cong \angle CBE$.
Prove: $\triangle ABE \cong \triangle DBC$.
14. *Given:* $\overline{AE} \cong \overline{BE}$, $\overline{ED} \cong \overline{EC}$, $\overline{DE} \perp \overline{AE}$, $\overline{CE} \perp \overline{EB}$.
Prove: $\triangle AEC \cong \triangle BED$.
15. *Given:* $\overleftrightarrow{AB}$, $\overline{DA} \cong \overline{DE}$, $\angle ADE \cong \angle BDC$, $\angle DAE \cong \angle DEC$.
Prove: $\triangle DAB \cong \triangle DEC$.

7. Using Congruent Triangles to Prove Line Segments Congruent and Angles Congruent

To help us understand the following method of proving line segments congruent and angles congruent, let us recall that when two triangles are congruent their corresponding sides are congruent and their corresponding angles are congruent. Remember that in two congruent triangles, corresponding sides are found opposite congruent angles, and corresponding angles are found opposite congruent sides.

To prove that two line segments are congruent or two angles are congruent:

1. Choose two triangles in which the line segments to be proved congruent are sides of the triangles or in which the angles to be proved congruent are angles of the triangles.
2. Prove by any appropriate method that the triangles are congruent.
3. Show that the line segments or angles to be proved congruent are corresponding parts of the triangles proved congruent and must, therefore, be congruent.

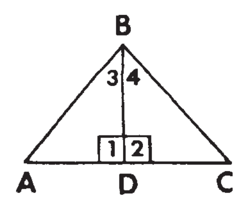
MODEL PROBLEMS

1. Prove that in $\triangle ABC$ if $\overline{BD}$ bisects $\angle ABC$, and $\overline{BD} \perp \overline{AC}$, then $\overline{BD}$ bisects $\overline{AC}$.

Given: In $\triangle ABC$, $\overline{BD}$ bisects $\angle ABC$.
$\overline{BD} \perp \overline{AC}$.

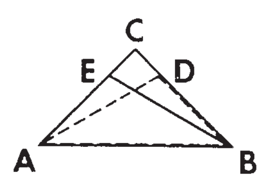
To prove: $\overline{AD} \cong \overline{CD}$.

Plan: To prove that $\overline{AD} \cong \overline{CD}$, show that the triangles which contain these lines, $\triangle ABD$ and $\triangle CBD$, are congruent, and that $\overline{AD}$ and $\overline{CD}$ are corresponding sides of these triangles.



Proof: *Statements*	*Reasons*
1. $\overline{BD}$ bisects $\angle ABC$.	1. Given.
2. $\angle 3 \cong \angle 4$. (a. $\cong$ a.)	2. The bisector of an angle divides the angle into two congruent angles.
3. $\overline{BD} \perp \overline{AC}$.	3. Given.
4. $\angle 1$ and $\angle 2$ are right angles.	4. Perpendicular lines intersect and form right angles.
5. $\angle 1 \cong \angle 2$. (a. $\cong$ a.)	5. If two angles are right angles, they are congruent.
6. $\overline{BD} \cong \overline{BD}$. (s. $\cong$ s.)	6. Reflexive property of congruence.
7. $\triangle ABD \cong \triangle CBD$.	7. a.s.a. $\cong$ a.s.a.
8. $\overline{AD} \cong \overline{CD}$. ($\overline{AD}$ is opposite $\angle 3$, and $\overline{CD}$ is opposite the congruent $\angle 4$.)	8. Corresponding parts of congruent triangles are congruent.

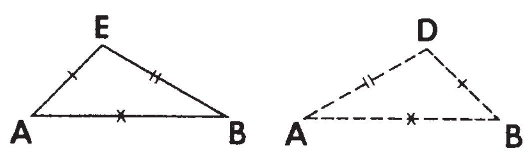
2. *a.* *Given:* $\overline{CA} \cong \overline{CB}$, $\overline{CE} \cong \overline{CD}$, $\overline{BE} \cong \overline{AD}$.



To prove: $\angle EAB \cong \angle DBA$.

Plan: To prove that $\angle EAB \cong \angle DBA$, show that the overlapping triangles EAB and DBA are congruent, and that $\angle EAB$ and $\angle DBA$ are corresponding angles of these congruent triangles.

Separate the Triangles



Proof: *Statements*	*Reasons*
1. $\overline{BE} \cong \overline{AD}$. (s. $\cong$ s.)	1. Given.
2. $\overline{CA} \cong \overline{CB}$, $\overline{CE} \cong \overline{CD}$.	2. Given.
3. $\overline{CA} - \overline{CE} \cong \overline{CB} - \overline{CD}$, or $\overline{AE} \cong \overline{BD}$. (s. $\cong$ s.)	3. If congruent segments are subtracted from congruent segments, the differences are congruent segments.

[The proof is continued on the next page.]

Proof: *Statements*	*Reasons*
4. $\overline{AB} \cong \overline{AB}$. (s. $\cong$ s.)	4. Reflexive property of congruence.
5. $\triangle EAB \cong \triangle DBA$.	5. s.s.s. $\cong$ s.s.s.
6. $\angle EAB \cong \angle DBA$. ($\angle EAB$ is opposite side $\overline{EB}$, and $\angle DBA$ is opposite the congruent side $\overline{DA}$.)	6. Corresponding parts of congruent triangles are congruent.

b. Using the results found in part *a*, find the measure of $\angle EAB$ if the measure of $\angle EAB$ is represented by $5x - 8$ and the measure of $\angle DBA$ is represented by $3x + 12$.

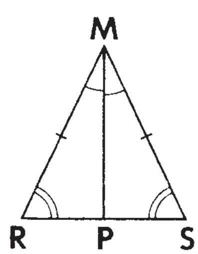
Solution:

1. $\angle EAB \cong \angle DBA$, or [Proved in part *a*.]
2. $m\angle EAB = m\angle DBA$
3. $5x - 8 = 3x + 12$
4. $5x - 3x = 12 + 8$
5. $2x = 20$
6. $x = 10$
7. $m\angle EAB = 5x - 8 = 5(10) - 8 = 50 - 8 = 42$.

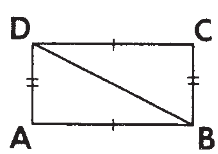
Answer: $m\angle EAB = 42$.

EXERCISES

In 1–6, the straight-line figures have been marked to indicate the pairs of congruent angles and pairs of congruent segments. (*a*) Name two triangles that are congruent and state the reason why the triangles are congruent. (*b*) In these triangles, name three additional pairs of parts that are congruent because they are corresponding parts of congruent triangles.



1.



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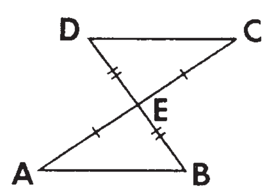

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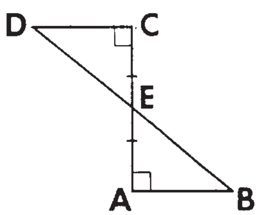

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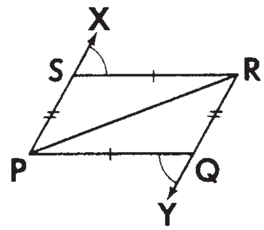

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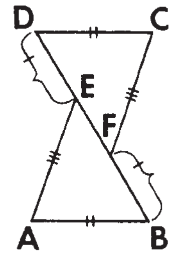

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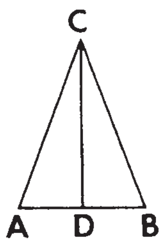

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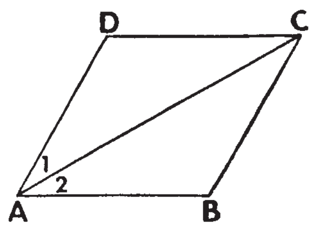

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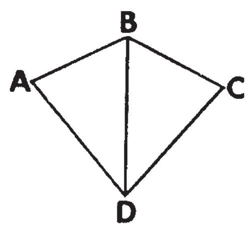

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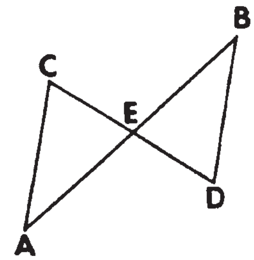


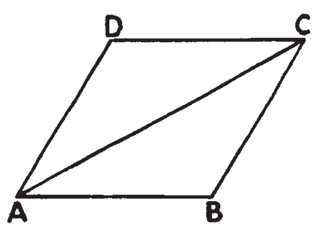
Ex. 10

7. $\overline{CA} \cong \overline{CB}$ and $\overline{AD} \cong \overline{BD}$. (*a*) Prove $\triangle ADC \cong \triangle BDC$. (*b*) Find three pairs of congruent angles in $\triangle ADC$ and $\triangle BDC$.

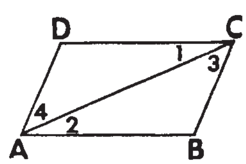
8. $\overline{AD} \cong \overline{AB}$ and $\angle 1 \cong \angle 2$. (*a*) Prove $\triangle ADC \cong \triangle ABC$. (*b*) Find three more pairs of congruent parts in $\triangle ADC$ and $\triangle ABC$.

9. $\overline{BD}$ bisects $\angle ABC$, and $\overrightarrow{DB}$ bisects $\angle ADC$. (*a*) Prove that $\triangle ABD \cong \triangle CBD$. (*b*) Find three more pairs of congruent parts in $\triangle ABD$ and $\triangle CBD$.

10. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are straight lines, $\overline{AE} \cong \overline{BE}$ and $\overline{CE} \cong \overline{DE}$, prove that $\angle C \cong \angle D$.



Ex. 11



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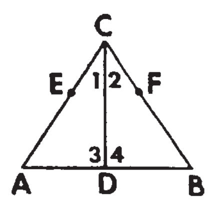
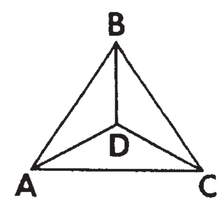


Ex. 13



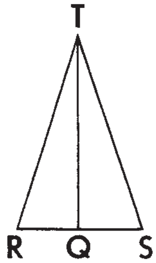
Ex. 14

11. If $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$, prove that $\angle B \cong \angle D$.

12. If $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$, prove that $\overline{DC} \cong \overline{BA}$.

13. If $\overleftrightarrow{CA}$ and $\overleftrightarrow{CB}$ are straight lines, $\angle 1 \cong \angle 2$, $\overline{CE} \cong \overline{CF}$, and $\overline{EA} \cong \overline{FB}$, prove that $\angle 3 \cong \angle 4$.

14. If $\overline{BA} \cong \overline{BC}$ and $\overline{DA} \cong \overline{DC}$, prove that $\overline{BD}$ bisects $\angle ABC$.



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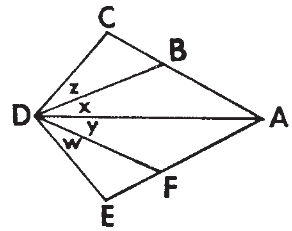

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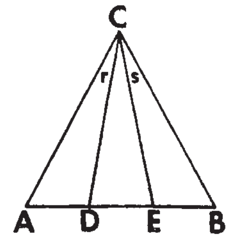
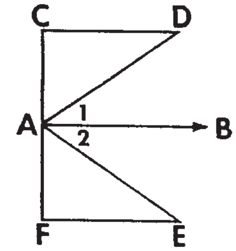


Ex. 17



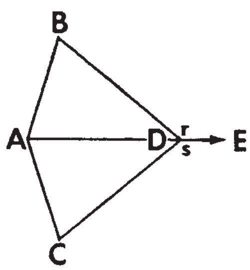
Ex. 18

15. If $\overline{TQ}$ bisects $\angle RTS$ and $\overline{TQ} \perp \overline{RS}$, prove that $\overline{TQ}$ bisects $\overline{RS}$.

16. If $\overline{DC} \cong \overline{DE}$, $\angle x \cong \angle y$, and $\angle z \cong \angle w$, prove that $\overline{AE} \cong \overline{AC}$.

17. If $\overleftrightarrow{AB}$ is a straight line, $\overline{AC} \cong \overline{BC}$, $\overline{CE} \cong \overline{CD}$, and $\overline{AE} \cong \overline{BD}$, prove that $\angle r \cong \angle s$.

18. If $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{CF}$, $\angle 1 \cong \angle 2$, and $\overline{AD} \cong \overline{AE}$, prove that $\angle D \cong \angle E$.



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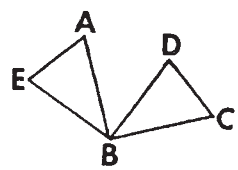

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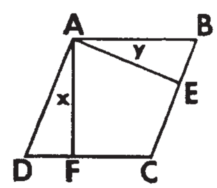

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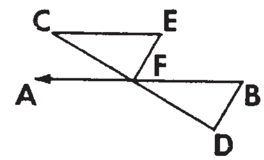


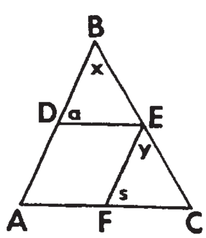
Ex. 22

19. If $\overleftrightarrow{AE}$ is a straight line, $\angle r \cong \angle s$, and $\overline{BD} \cong \overline{CD}$, prove that $\overline{AB} \cong \overline{AC}$.

20. If $\overline{AB} \perp \overline{BC}$, $\overline{DB} \perp \overline{BE}$, $\overline{BC} \cong \overline{BE}$, and $\overline{DB} \cong \overline{AB}$, prove that $\angle D \cong \angle A$.

21. If $\angle D$ and $\angle B$ are both supplementary to $\angle C$, $\overline{AD} \cong \overline{AB}$, and $\overline{DF} \cong \overline{BE}$, prove that $\angle x \cong \angle y$.

22. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are straight lines, $\angle ECF \cong \angle CFA$, $\overline{CF} \cong \overline{FD}$, and $\overline{CE} \cong \overline{FB}$, prove that $\angle E \cong \angle B$.



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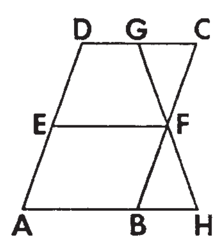

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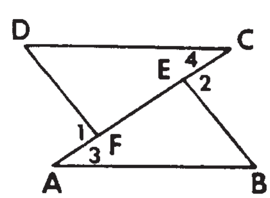

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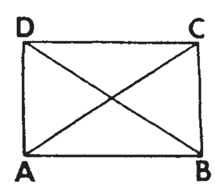


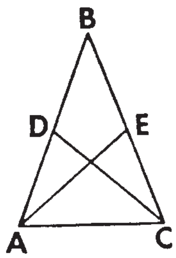
Ex. 26

23. In triangle ABC, D is the midpoint of $\overline{AB}$, E is the midpoint of $\overline{BC}$, $\angle x \cong \angle y$, and $\overline{AD} \cong \overline{EF}$. Prove that $\angle a \cong \angle s$.

24. If F is the midpoint of $\overline{GH}$, E is the midpoint of $\overline{AD}$, $\overline{FC} \cong \overline{ED}$, $\overline{BF} \cong \overline{AE}$, and $\overleftrightarrow{BC}$ is a straight line, prove that $\overline{GC} \cong \overline{HB}$.

25. If $\overleftrightarrow{AC}$ is a straight line, $\overline{AF} \cong \overline{CE}$, $\angle 3 \cong \angle 4$, and $\angle 1 \cong \angle 2$, prove that $\angle B \cong \angle D$.

26. If $\overline{DA} \perp \overline{AB}$, $\overline{CB} \perp \overline{AB}$, and $\overline{AD} \cong \overline{BC}$, prove that $\overline{AC} \cong \overline{BD}$.



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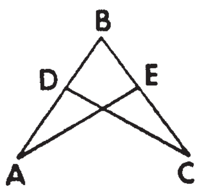

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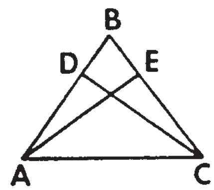

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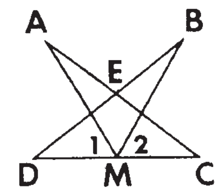


Ex. 30

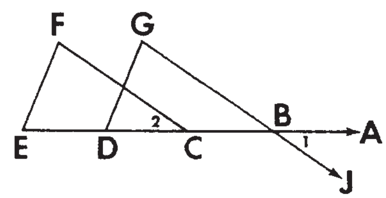
27. If $\overline{DA} \cong \overline{EC}$ and $\overline{AE} \cong \overline{CD}$, prove that $\angle EAC \cong \angle DCA$.

28. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are straight lines, $\overline{BD} \cong \overline{BE}$, and $\overline{DA} \cong \overline{EC}$, prove that $\angle A \cong \angle C$.

29. If $\overline{CD} \perp \overline{AB}$, $\overline{AE} \perp \overline{BC}$, and $\overline{BD} \cong \overline{BE}$, prove that $\overline{AE} \cong \overline{CD}$.

30. If $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are straight lines which intersect at E, $\angle D \cong \angle C$, $\angle 2 \cong \angle 1$, and M is the midpoint of $\overline{DC}$, prove that $\overline{DB} \cong \overline{CA}$.

31. If $\overleftrightarrow{EA}$ and $\overleftrightarrow{GJ}$ are straight lines, $\overline{ED} \cong \overline{BC}$, $\angle 1 \cong \angle 2$, and $\overline{FC} \cong \overline{GB}$, prove that $\angle F \cong \angle G$.



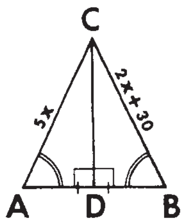
Ex. 31

32. In isosceles triangle ABC, the congruent sides $\overline{CA}$ and $\overline{CB}$ are extended through A and B to points D and E respectively so that $\overline{AD} \cong \overline{BE}$. $\overline{AE}$ and $\overline{BD}$ are drawn. Prove that triangles CDB and CEA are congruent and $\angle D \cong \angle E$.

33. Triangle ABC is congruent to triangle $A'B'C'$. If $m\angle C$ is represented by $2x - 10$ and $m\angle C'$ is represented by $x + 30$: (*a*) Find x. (*b*) Find $m\angle C$. (*c*) Find $m\angle B$ if it is represented by $x - 25$.

34. Triangle DEF is congruent to triangle $D'E'F'$. If EF is represented by $3x + 2$, $E'F'$ is represented by $x + 10$, and ED is represented by $x + 2$, find x, ED, and $E'D'$.

In 35–37, the straight-line figures have been marked to indicate pairs of congruent angles and pairs of congruent sides: (*a*) Prove two triangles congruent. (*b*) Find the value of x. (*c*) Find the measure of each side or angle which is represented in terms of x.



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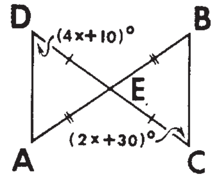

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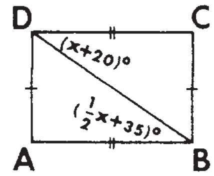
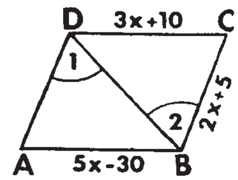


37.

38. *Given:* $\overline{AD} \cong \overline{CB}$ and $\angle 1 \cong \angle 2$. (*a*) *Prove:* $\triangle ADB \cong \triangle CBD$. (*b*) *Prove:* $\overline{AB} \cong \overline{CD}$. (*c*) Find the length of $\overline{AB}$ and of $\overline{CD}$. (*d*) Find the length of $\overline{CB}$ and of $\overline{AD}$.



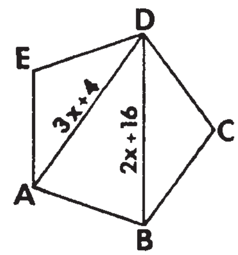
Ex. 38

39. If all sides of polygon $ABCDE$ are congruent and $\angle E \cong \angle C$, find AD and BD.

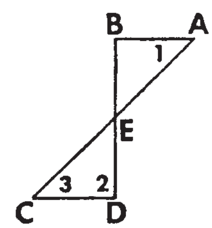
40. Triangle ABC is congruent to triangle $A'B'C'$. If AB is represented by $2x + y$, $A'B' = 7$, $BC = 11$, and $B'C'$ is represented by $4x + y$, find x and y.

41. Triangle ABC is congruent to triangle $A'B'C'$. If $m\angle A$ is represented by $x + 10$, $m\angle A'$ is represented by $y + 20$, $m\angle B$ is represented by $3x$, and $m\angle B'$ is represented by $x + 3y$, find $m\angle A$ and $m\angle B$.

42. If $\overline{AB} \perp \overline{BD}$, $\overline{CD} \perp \overline{BD}$, and $\overleftrightarrow{AC}$ bisects $\overline{BD}$, find the number of degrees in $\angle 1$ and $\angle 3$ when $m\angle 1$ is represented by $2y + 20$, $m\angle 2$ is represented by $2x + 3y$, and $m\angle 3$ is represented by $x + y$.



Ex. 39



Ex. 42

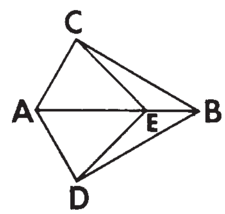
8. Using Two Pairs of Congruent Triangles

Sometimes it is impossible to use the *given* in order to prove directly that a particular pair of triangles is congruent. In such cases, the *given* may contain enough information to first prove another pair of triangles congruent. Then corresponding congruent parts in these congruent triangles may be used to prove the original pair of triangles congruent. See how this is done in the following example:

MODEL PROBLEM

Given: $\overleftrightarrow{AB}$ is a straight line.
$\overline{AC} \cong \overline{AD}$.
$\overline{BC} \cong \overline{BD}$.

To prove: $\overline{CE} \cong \overline{DE}$.



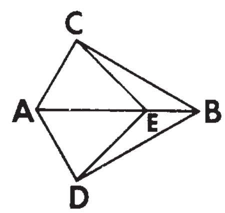
We can prove $\overline{CE} \cong \overline{DE}$ if we can prove that $\triangle ACE \cong \triangle ADE$. In these two triangles, we know only that $\overline{AC} \cong \overline{AD}$ (given) and $\overline{AE} \cong \overline{AE}$ (reflexive property of congruence). If we could prove $\angle CAE \cong \angle DAE$, then triangles ACE and ADE would be congruent by s.a.s. $\cong$ s.a.s. We can prove $\angle CAE \cong \angle DAE$ by showing that $\triangle CAB$ and $\triangle DAB$, which also contain these angles, are congruent.

In $\triangle CAB$ and $\triangle DAB$, $\overline{AC} \cong \overline{AD}$ (given), $\overline{BC} \cong \overline{BD}$ (given), and $\overline{AB} \cong \overline{AB}$ (reflexive property of congruence). Therefore, $\triangle CAB \cong \triangle DAB$ by s.s.s $\cong$ s.s.s. $\angle CAB$, or $\angle CAE \cong \angle DAB$, or $\angle DAE$, because they are corresponding parts of congruent $\triangle CAB$ and $\triangle DAB$.

Now having proved that $\angle CAE \cong \angle DAE$, we can prove that $\triangle ACE \cong \triangle ADE$ by s.a.s. $\cong$ s.a.s. Hence, $\overline{CE} \cong \overline{DE}$ because they are corresponding sides of these congruent triangles. The formal proof follows:

Given: $\overleftrightarrow{AB}$ is a straight line.
$\overline{AC} \cong \overline{AD}$.
$\overline{BC} \cong \overline{BD}$.

To prove: $\overline{CE} \cong \overline{DE}$.



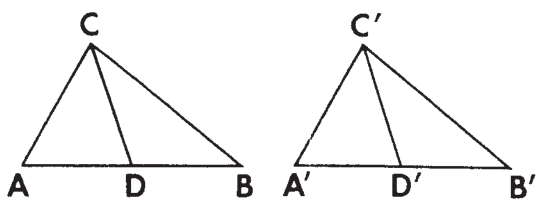
Proof:

Statements	*Reasons*
1. In $\triangle CAB$ and $\triangle DAB$, $\overline{AC} \cong \overline{AD}$. (s. $\cong$ s.)	1. Given.
2. $\overline{BC} \cong \overline{BD}$. (s. $\cong$ s.)	2. Given.
3. $\overline{AB} \cong \overline{AB}$. (s. $\cong$ s.)	3. Reflexive property of congruence.
4. $\triangle CAB \cong \triangle DAB$.	4. s.s.s. $\cong$ s.s.s.
5. $\angle CAB \cong \angle DAB$, or $\angle CAE \cong \angle DAE$.	5. Corresponding parts of congruent triangles are congruent.
6. In $\triangle ACE$ and $\triangle ADE$, $\angle CAE \cong \angle DAE$. (a. $\cong$ a.)	6. Proved in steps 1–5.

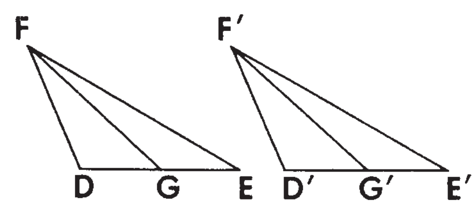
[The proof is continued on the next page.]

Proof: *Statements*		*Reasons*
7. $\overline{AC} \cong \overline{AD}$.	(s. ≅ s.)	7. Given.
8. $\overline{AE} \cong \overline{AE}$.	(s. ≅ s.)	8. Reflexive property of congruence.
9. $\triangle ACE \cong \triangle ADE$.		9. s.a.s. ≅ s.a.s.
10. $\overline{CE} \cong \overline{DE}$.		10. Corresponding parts of congruent triangles are congruent.

EXERCISES

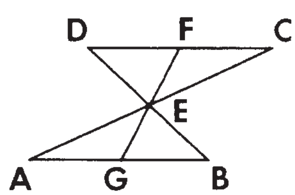


Ex. 1

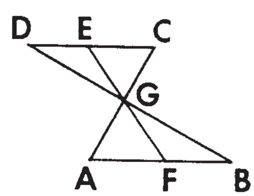


Ex. 2

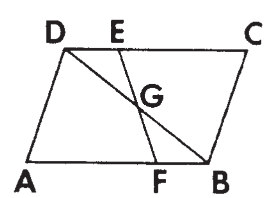
1. *Given:* $\triangle ABC \cong \triangle A'B'C'$, $\overline{CD}$ bisects $\angle C$, $\overline{C'D'}$ bisects $\angle C'$.
   Prove: $\overline{CD} \cong \overline{C'D'}$.
2. *Given:* $\triangle DEF \cong \triangle D'E'F'$, $\overline{FG}$ and $\overline{F'G'}$ are medians.
   Prove: $\overline{FG} \cong \overline{F'G'}$.



Ex. 3

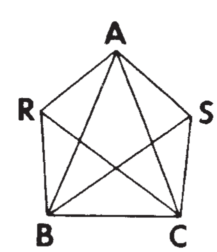


Ex. 4

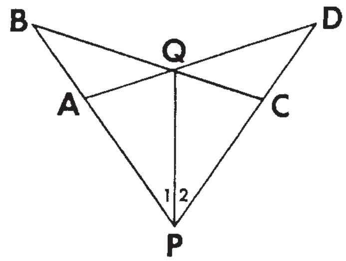


Ex. 5

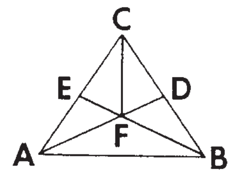
3. *Given:* $\overleftrightarrow{AC}$, $\overleftrightarrow{BD}$, and $\overleftrightarrow{GF}$ are straight lines, $\overline{AE} \cong \overline{CE}$, $\overline{FE} \cong \overline{GE}$.
   Prove: (*a*) $\angle C \cong \angle A$. (*b*) $\overline{DC} \cong \overline{BA}$.
4. *Given:* $\overleftrightarrow{EF}$, $\overline{AC}$ and $\overline{BD}$ bisect each other at G.
   Prove: (*a*) $\angle D \cong \angle B$. (*b*) $\overline{GE} \cong \overline{GF}$.
5. *Given:* $\overline{AD} \cong \overline{CB}$, $\overline{DC} \cong \overline{BA}$, $\overline{EF}$ bisects $\overline{BD}$ at G.
   Prove: (*a*) $\angle CDB \cong \angle ABD$. (*b*) $\overline{GE} \cong \overline{GF}$.
6. *Given:* $\overline{AB} \cong \overline{AC}$, $\angle RAB \cong \angle SAC$, and $\angle RBA \cong \angle SCA$.
   Prove: (*a*) $\overline{AR} \cong \overline{AS}$. (*b*) $\angle ACR \cong \angle ABS$.



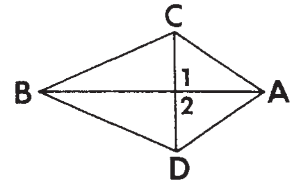
Ex. 6



Ex. 7



Ex. 8

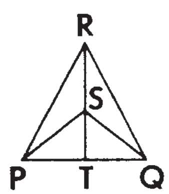


Ex. 9

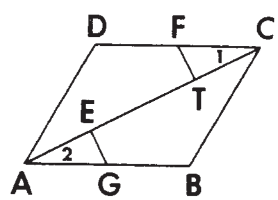
7. *Given:* $\overleftrightarrow{PQ}$, $\overleftrightarrow{PAB}$, $\overleftrightarrow{PCD}$, $\overleftrightarrow{AQD}$ and $\overleftrightarrow{CQB}$. $\angle 1 \cong \angle 2$ and $\overline{AP} \cong \overline{CP}$.
Prove: (*a*) $\triangle APQ \cong \triangle CPQ$. (*b*) $\overline{QB} \cong \overline{QD}$.

8. *Given:* $\overline{EF} \cong \overline{DF}$, and $\angle EFC \cong \angle DFC$.
Prove: (*a*) $\angle ECF \cong \angle DCF$. (*b*) $\overline{AC} \cong \overline{BC}$.

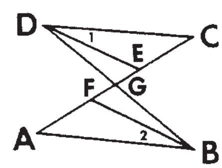
9. *Given:* $\overline{AC} \cong \overline{AD}$, $\overline{BC} \cong \overline{BD}$.
Prove: $\angle 1 \cong \angle 2$.



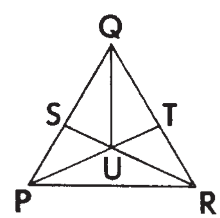
Ex. 10



Ex. 11



Ex. 12



Ex. 13

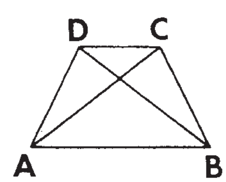
10. *Given:* $\overline{RP} \cong \overline{RQ}$, $\overline{SP} \cong \overline{SQ}$.
Prove: $\overline{RT}$ bisects $\overline{PQ}$.

11. *Given:* $\overleftrightarrow{AC}$, F is the midpoint of $\overline{DC}$, G is the midpoint of $\overline{AB}$, $\overline{FT} \perp \overline{AC}$, $\overline{GE} \perp \overline{AC}$, $\angle 1 \cong \angle 2$, $\overline{AT} \cong \overline{CE}$.
Prove: $\angle B \cong \angle D$.

12. *Given:* $\overline{AC}$ and $\overline{BD}$ bisect each other at G, $\angle 1 \cong \angle 2$.
Prove: $\overline{EC} \cong \overline{FA}$.

13. *Given:* $\overline{QP} \cong \overline{QR}$, $\overline{QS} \cong \overline{QT}$.
Prove: $\overline{QU}$ bisects $\angle PQR$.

14. *Given:* In quadrilateral $ABCD$, $\overline{AD} \cong \overline{BC}$, and $\angle BAD \cong \angle ABC$.
Prove: $\angle BCD \cong \angle ADC$.



Ex. 14

15. In quadrilateral $ABCD$, $\overline{AB} \cong \overline{AD}$, and $\overline{BC} \cong \overline{DC}$. Diagonal $\overline{AC}$ is extended through C to E and segments $\overline{BE}$ and $\overline{DE}$ are drawn.
Prove: $\overline{BE} \cong \overline{DE}$.

16. *Prove:* In two congruent triangles, two corresponding medians are congruent.

17. *Prove:* In two congruent triangles, two corresponding angle bisectors are congruent.

9. The Isosceles Triangle and the Equilateral Triangle

Properties of an Isosceles Triangle

***Theorem 9.* If two sides of a triangle are congruent, the angles opposite these sides are congruent.**

OR

The base angles of an isosceles triangle are congruent.

[The proof for this theorem appears on page 747.]

In $\triangle ABC$ (Fig. 3–16), if $\overline{AB} \cong \overline{AC}$, then $\angle C \cong \angle B$.

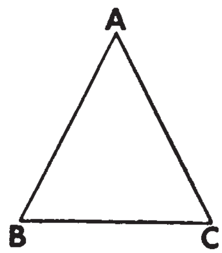


Fig. 3–16

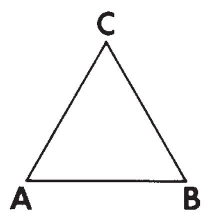


Fig. 3–17

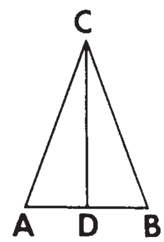


Fig. 3–18

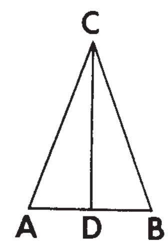


Fig. 3–19

***Corollary T9–1.* Every equilateral triangle is equiangular.**

If in $\triangle ABC$ (Fig. 3–17) $\overline{BC} \cong \overline{CA} \cong \overline{AB}$, then $\angle A \cong \angle B \cong \angle C$, or $\triangle ABC$ is equiangular.

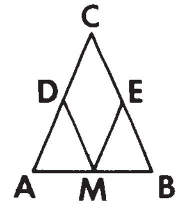
***Corollary T9–2.* The bisector of the vertex angle of an isosceles triangle bisects the base.**

If in $\triangle ABC$ (Fig. 3–18) $\overline{CA} \cong \overline{CB}$, and $\overline{CD}$ bisects $\angle ACB$ ($\angle ACD \cong \angle BCD$), then $\overline{CD}$ bisects side $\overline{AB}$ ($\overline{AD} \cong \overline{BD}$).

***Corollary T9–3.* The bisector of the vertex angle of an isosceles triangle is perpendicular to the base.**

If in $\triangle ABC$ (Fig. 3–19) $\overline{CA} \cong \overline{CB}$ and $\overline{CD}$ bisects $\angle ACB$ ($\angle ACD \cong \angle BCD$), then $\overline{CD} \perp \overline{AB}$.

MODEL PROBLEM



Given: Isosceles $\triangle ABC$ with $\overline{CA} \cong \overline{CB}$.
M is the midpoint of $\overline{AB}$. $\overline{AD} \cong \overline{BE}$.

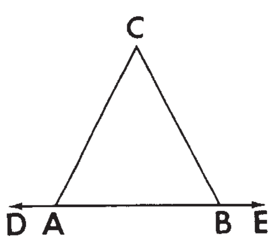
To prove: $\overline{MD} \cong \overline{ME}$.

Plan: To prove that $\overline{MD} \cong \overline{ME}$, prove that $\triangle ADM$ and $\triangle BEM$, which have $\overline{MD}$ and $\overline{ME}$ as corresponding sides, are congruent by s.a.s $\cong$ s.a.s.

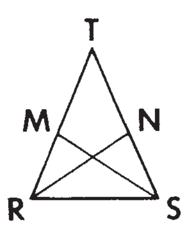
Proof: *Statements*	*Reasons*
1. $\overline{AD} \cong \overline{BE}$. (s. $\cong$ s.)	1. Given.
2. M is the midpoint of $\overline{AB}$.	2. Given.
3. $\overline{AM} \cong \overline{BM}$. (s. $\cong$ s.)	3. A midpoint divides a line segment into two congruent parts.
4. $\overline{CA} \cong \overline{CB}$.	4. Given.
5. $\angle A \cong \angle B$. (a. $\cong$ a.)	5. If two sides of a triangle are congruent, the angles opposite these sides are congruent.
6. $\triangle ADM \cong \triangle BEM$.	6. s.a.s. $\cong$ s.a.s.
7. $\overline{MD} \cong \overline{ME}$.	7. Corresponding parts of congruent triangles are congruent.

EXERCISES

1. In $\triangle ABC$, if $\overline{CA} \cong \overline{CB}$ and $m\angle A = 50$, find $m\angle B$.
2. In triangle ABC, $\overline{AB} \cong \overline{BC}$. If $AB = 5x$ and $BC = 2x + 18$, find AB and BC.
3. In isosceles triangle ABC, $\overline{AB} \cong \overline{BC}$. If $AB = 5x + 10$, $BC = 3x + 40$, and $AC = 2x + 30$, find the length of each side of the triangle.
4. In triangle ABC, $\overline{AB} \cong \overline{BC}$. If $m\angle A = 7x$ and $m\angle C = 2x + 50$, find $m\angle A$ and $m\angle C$.
5. In triangle EFG, $\overline{EF} \cong \overline{FG}$. If $m\angle E = 4x + 50$, $m\angle F = 2x + 60$, and $m\angle G = 14x + 30$, find $m\angle E$, $m\angle F$, and $m\angle G$.
6. $\overline{BD}$ is the bisector of vertex angle B of isosceles triangle ABC. If $AB = 2x + 2y$, $AD = x + 2y$, $DC = 4x$, and $BC = 10$, find AB, AD, and AC.



Ex. 7



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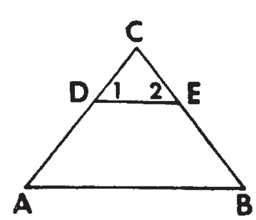
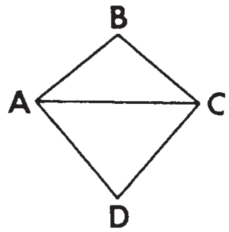


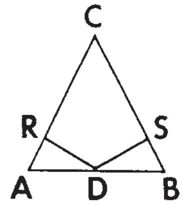
Ex. 9

7. In $\triangle ABC$, if $\overline{CA} \cong \overline{CB}$ and $\overleftrightarrow{DE}$ is a straight line, prove that $\angle CAD \cong \angle CBE$.

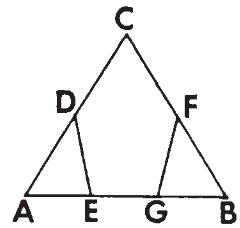
8. In $\triangle RST$, if $\overline{TR} \cong \overline{TS}$, $\overline{RN}$ bisects $\angle R$, and $\overline{SM}$ bisects $\angle S$, prove that $\angle NRS \cong \angle MSR$.
9. If $\overline{CA} \cong \overline{CB}$, and $\overline{DA} \cong \overline{EB}$, prove that $\angle 1 \cong \angle 2$.



Ex. 10

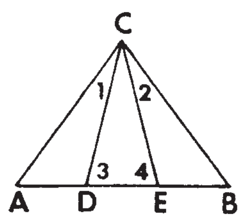


Ex. 11

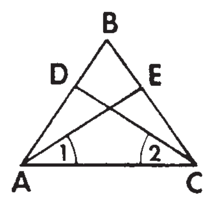


Ex. 12

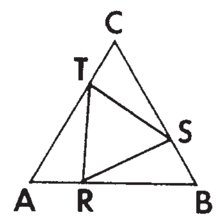
10. Isosceles triangles ABC and ADC have the common base $\overline{AC}$. Prove that $\angle BAD \cong \angle BCD$.
11. In $\triangle ABC$, if $\overline{CA} \cong \overline{CB}$, $\overline{AR} \cong \overline{BS}$, $\overline{DR} \perp \overline{AC}$, and $\overline{DS} \perp \overline{BC}$, prove that $\overline{DR} \cong \overline{DS}$.
12. In isosceles triangle ABC, D and F are the midpoints of the congruent legs. E and G are the trisection points of the base ($\overline{AE} \cong \overline{EG} \cong \overline{GB}$). Prove that $\overline{DE} \cong \overline{FG}$.



Ex. 13

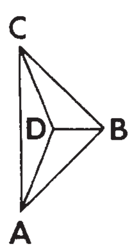


Ex. 14

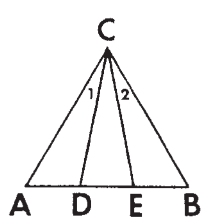


Ex. 15

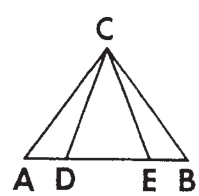
13. If $\overleftrightarrow{AB}$ is a straight line, $\overline{CA} \cong \overline{CB}$, and $\angle 1 \cong \angle 2$, prove that $\angle 3 \cong \angle 4$.
14. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are straight lines, $\overline{BD} \cong \overline{BE}$, and $\overline{DA} \cong \overline{EC}$, prove that $\angle 1 \cong \angle 2$.
15. If $\triangle ABC$ is an equilateral triangle and $\overline{CT} \cong \overline{AR} \cong \overline{BS}$, prove that $\overline{TR} \cong \overline{RS} \cong \overline{ST}$.



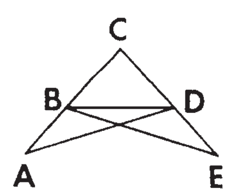
Ex. 16



Ex. 17



Ex. 18

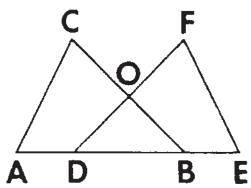


Ex. 19

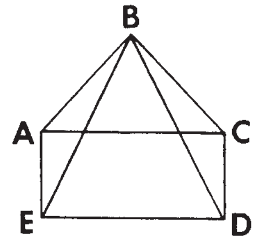
16. In $\triangle ABC$, if $\overline{AB} \cong \overline{BC}$ and $\overline{DB}$ bisects $\angle B$, prove that $\angle ACD \cong \angle CAD$.
17. If in $\triangle ABC$, $\overline{CA} \cong \overline{CB}$ and $\angle 1 \cong \angle 2$, prove that $\overline{AE} \cong \overline{BD}$.

18. If $\overline{AD} \cong \overline{EB}$, $\overline{CD} \cong \overline{CE}$, and $\overleftrightarrow{AB}$ is a straight line, prove that $\overline{AC} \cong \overline{BC}$.

19. If $\overline{CB} \cong \overline{CD}$, $\overline{BA} \cong \overline{DE}$, and $\overleftrightarrow{CA}$ and $\overleftrightarrow{CE}$ are straight lines, prove that $\angle EBD \cong \angle ADB$.



Ex. 20



Ex. 21

20. *Given:* On line $\overleftrightarrow{AE}$, $\overline{AD} \cong \overline{BE}$. $\overline{DF}$ and $\overline{BC}$ are congruent and bisect each other at O.
Prove: $\overline{AC} \cong \overline{EF}$.

21. In the figure, $EDCA$ is a quadrilateral, $\overline{EA} \cong \overline{DC}$, $\overline{BA} \cong \overline{BC}$, $\overline{EA} \perp \overline{AC}$, and $\overline{DC} \perp \overline{AC}$.
Prove: $\angle BED \cong \angle BDE$.

22. *Prove:* The line segments joining the midpoint of the base of an isosceles triangle to the midpoints of the legs are congruent.

23. *Prove:* The bisectors of the base angles of an isosceles triangle are congruent.

24. *Prove:* The medians to the legs of an isosceles triangle are congruent.

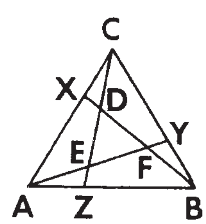
25. In isosceles triangle DEF, $\overline{ED} \cong \overline{EF}$. Point P is chosen between points E and D; point Q is chosen between points E and F, so that $\overline{EP} \cong \overline{EQ}$. Segments $\overline{DQ}$ and $\overline{FP}$ are drawn. *Prove:* $\triangle PDF \cong \triangle QFD$.

26. O is the midpoint of base $\overline{AB}$ of isosceles triangle ABC. $\overline{AC}$ and $\overline{BC}$ are extended through C to points E and D respectively so that $\overline{CE}$ is congruent to $\overline{CD}$. Line segments $\overline{DO}$ and $\overline{EO}$ are drawn. *Prove:* $\overline{DO} \cong \overline{EO}$.

27. In triangle CDE, $\overline{CD} \cong \overline{CE}$. R is the midpoint of $\overline{CD}$ and S is the midpoint of $\overline{CE}$. $\overleftrightarrow{RM}$ is drawn perpendicular to $\overleftrightarrow{CD}$; $\overleftrightarrow{SN}$ is drawn perpendicular to $\overleftrightarrow{CE}$, M and N being points on $\overline{DE}$ or $\overline{DE}$ extended. Prove that $\overline{RM} \cong \overline{SN}$.

28. In isosceles triangle ABC, $\overline{CA} \cong \overline{CB}$. D is a point on $\overline{CA}$ and E is a point on $\overline{CB}$. $\overline{AD} \cong \overline{BE}$. $\overline{BD}$ is drawn and extended its own length through D to X, $\overline{AE}$ is drawn and extended its own length through E to Y, and $\overline{XA}$ and $\overline{YB}$ are drawn. *Prove:* (*a*) $\overline{BD} \cong \overline{AE}$. (*b*) $\overline{XA} \cong \overline{YB}$.

29. In equilateral triangle ABC, if $\overline{CX} \cong \overline{AZ} \cong \overline{BY}$, prove that $\overline{DE} \cong \overline{EF} \cong \overline{FD}$.



Ex. 29

10. Proving a Triangle Isosceles

The Converse of a Statement

The *converse* of a given statement is another statement which is formed by interchanging the hypothesis and the conclusion in the given statement. For example:

1. *Given Statement:*	If a man drives a Cadillac, then he drives an American car.
Converse of Statement:	If a man drives an American car, then he drives a Cadillac.
2. *Given Statement:*	If three sides of one triangle are congruent respectively to three sides of another triangle, then the triangles are congruent.
Converse of Statement:	If two triangles are congruent, then three sides of one triangle are congruent respectively to three sides of the other triangle.

In general, for a given conditional statement, we have the following:

Given Statement:	If p, then q.
Converse of Statement:	If q, then p.

Observe that in example 1, the original statement is true but the converse of the original statement is false. Notice, too, that in example 2, the original statement is true and the converse of the original statement is also true. Hence, we see that the converse of a true statement is not necessarily a true statement. We cannot assume that the converse of a true statement is true. We must prove it.

We will discuss the topic of converses in greater detail in the chapter dealing with "Improvement of Reasoning."

The Converse of the Isosceles Triangle Theorem

We have proved the isosceles triangle theorem, "If two sides of a triangle are congruent, the angles opposite these sides are congruent." We cannot assume that the converse of this theorem, whose statement follows, is true. We must prove it.

Theorem 10. **If two angles of a triangle are congruent, the sides opposite these angles are congruent.**

OR

If two angles of a triangle are congruent, the triangle is isosceles.

[A proof for this theorem appears on pages 748–749. An alternate proof appears on page 751.]

In $\triangle ABC$ (Fig. 3–20), if $\angle A \cong \angle B$, then $\overline{BC} \cong \overline{AC}$, or $\triangle ABC$ is isosceles.

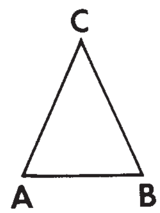


Fig. 3–20

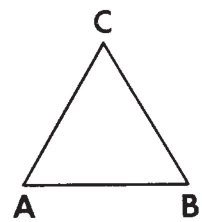


Fig. 3–21

Corollary T10–1. **If a triangle is equiangular, it is equilateral.**

In $\triangle ABC$ (Fig. 3–21), if $\angle A \cong \angle B \cong \angle C$, then $\overline{BC} \cong \overline{CA} \cong \overline{AB}$, or $\triangle ABC$ is equilateral.

To prove that a triangle is isosceles, prove that one of the following statements is true:

1. Two sides of the triangle are congruent.
2. Two angles of the triangle are congruent.

Sentences Involving "If and Only If"

It is possible to write a statement and its converse in one sentence. We can do this by inserting the expression "if and only if" between the hypothesis and the conclusion. For example, Theorem 9 (page 136) and Theorem 10 (page 140) can be combined as follows:

Two sides of a triangle are congruent if and only if the angles opposite these sides are congruent.

This theorem is a short way of stating the following two theorems:

Theorem 9. **If two sides of a triangle are congruent, then the angles opposite these sides are congruent.**

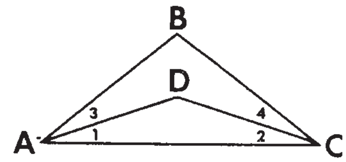
Theorem 10. **If two angles of a triangle are congruent, then the sides opposite these angles are congruent.**

If we wish to prove a statement of the form "p if and only if q," where p and q are statements, we must prove the following two statements:

1. If p, then q.
2. If q, then p.

MODEL PROBLEM

Given: In $\triangle ABC$, $\overline{BA} \cong \overline{BC}$.
$\overline{AD}$ bisects $\angle BAC$.
$\overline{CD}$ bisects $\angle BCA$.



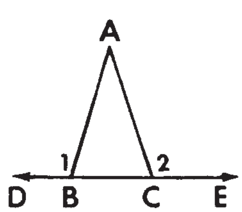
To prove: $\triangle ADC$ is an isosceles triangle.

Plan: In order to prove that $\triangle ADC$ is isosceles, prove that two of its sides, $\overline{CD}$ and $\overline{AD}$, are congruent. In order to prove $\overline{CD} \cong \overline{AD}$, prove that in $\triangle ADC$ the angles opposite these sides, $\angle 1$ and $\angle 2$, are congruent.

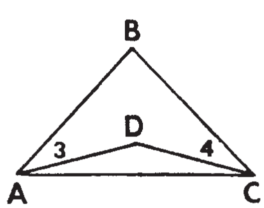
Proof:

Statements	*Reasons*
1. $\overline{BA} \cong \overline{BC}$.	1. Given.
2. $\angle BAC \cong \angle BCA$.	2. If two sides of a triangle are congruent, the angles opposite these sides are congruent.
3. $\overline{AD}$ bisects $\angle BAC$. $\overline{CD}$ bisects $\angle BCA$.	3. Given.
4. $\angle 1 \cong \angle 3$, or $m\angle 1 = \frac{1}{2}m\angle BAC$. $\angle 2 \cong \angle 4$, or $m\angle 2 = \frac{1}{2}m\angle BCA$.	4. The bisector of an angle divides the angle into two angles whose measures are equal.
5. $m\angle 1 = m\angle 2$, or $\angle 1 \cong \angle 2$.	5. Halves of equal quantities are equal.
6. $\overline{CD} \cong \overline{AD}$.	6. If two angles of a triangle are congruent, the sides opposite these angles are congruent.
7. $\triangle ADC$ is an isosceles triangle.	7. If a triangle has two congruent sides, it is an isosceles triangle.

EXERCISES



Ex. 1



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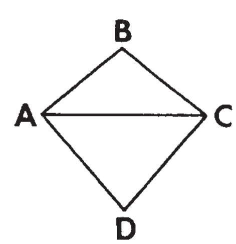
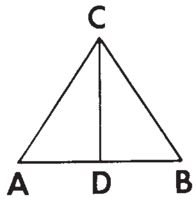


Ex. 3

1. If $\angle 1 \cong \angle 2$ and $\overleftrightarrow{DE}$ is a straight line, prove that $\triangle ABC$ is an isosceles triangle.
2. If $\overline{AB} \cong \overline{BC}$ and $\angle 3 \cong \angle 4$, prove that $\triangle ADC$ is an isosceles triangle.
3. If $\overline{BA} \cong \overline{BC}$, $\overline{DA} \perp \overline{AB}$, and $\overline{DC} \perp \overline{CB}$, prove that $\triangle ADC$ is an isosceles triangle.



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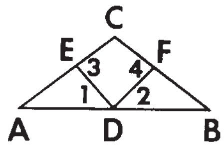
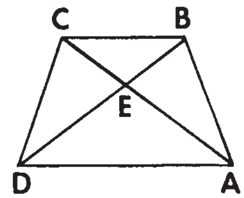
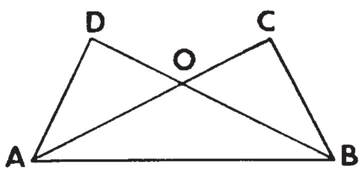


Ex. 5



Ex. 6

4. If $\overline{CD} \perp \overline{AB}$, and $\overline{CD}$ bisects $\overline{AB}$, prove that $\triangle ABC$ is an isosceles triangle.
5. If $\angle 1 \cong \angle 2$, $\overline{DE} \cong \overline{DF}$, and $\angle 3 \cong \angle 4$, prove that $\triangle ABC$ is an isosceles triangle.
6. If $\overline{AB} \cong \overline{DC}$, and $\overline{AC} \cong \overline{DB}$, prove that $\triangle DEA$ is an isosceles triangle.



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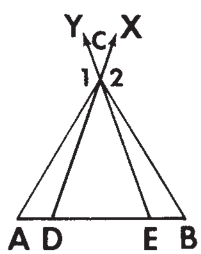
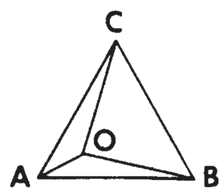
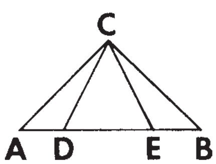


Ex. 8

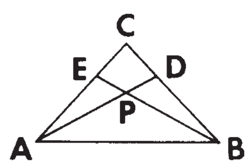


Ex. 9

7. In $\triangle ABC$ and $\triangle ABD$, if $\overline{DA} \cong \overline{CB}$ and $\angle DAB \cong \angle CBA$, prove that $\triangle AOB$ is an isosceles triangle.
8. $\overline{AC} \cong \overline{BC}$; $\overleftrightarrow{AB}$, $\overleftrightarrow{EY}$, and $\overleftrightarrow{DX}$ are straight lines; and $\angle 1 \cong \angle 2$. Prove that $\triangle DCE$ is an isosceles triangle.
9. *Given:* In $\triangle ABC$, $\overline{AB} \cong \overline{BC} \cong \overline{CA}$, and $\angle OCB \cong \angle OBC$.
   Prove: $\overline{AO}$ bisects $\angle CAB$.



Ex. 10



Ex. 11

10. *Given:* $\overleftrightarrow{AB}$, $\overline{AD} \cong \overline{BE}$, $\angle CDE \cong \angle CED$.
    Prove: $\triangle ACB$ is an isosceles triangle.

11. *Given:* $\triangle ACB$ with $\overline{CA} \cong \overline{CB}$, and $\overline{AD}$ and $\overline{BE}$ intersecting at P so that $\angle PAB \cong \angle PBA$.
Prove: $\overline{PE} \cong \overline{PD}$.

12. In isosceles triangle ABC, $\overline{AB} \cong \overline{BC}$. Points D and E are taken on $\overline{AB}$ and $\overline{BC}$ respectively so that $\overline{BD} \cong \overline{BE}$. $\overline{AE}$ and $\overline{CD}$ are drawn and intersect at H. *Prove:* (*a*) $\overline{AE} \cong \overline{CD}$. (*b*) $\triangle AHC$ is an isosceles triangle.

13. *Prove:* If a triangle is equiangular, it is equilateral.

11. Proving Lines Perpendicular

We have learned that two lines are perpendicular if they intersect and form right angles. Since a definition is reversible, if line $\overleftrightarrow{MN}$ and line $\overleftrightarrow{AB}$ intersect forming two right angles, $\angle r$ and $\angle s$, then line $\overleftrightarrow{MN}$ is perpendicular to line $\overleftrightarrow{AB}$. (See Fig. 3–22.)

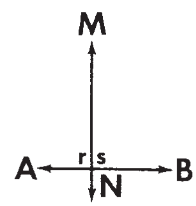


Fig. 3–22

If lines $\overleftrightarrow{MN}$ and $\overleftrightarrow{AB}$ intersect forming two congruent adjacent angles, $\angle r$ and $\angle s$, then since $m\angle r + m\angle s = 180$, we can show that $m\angle r = 90$ and $m\angle s = 90$, or that both $\angle r$ and $\angle s$ are right angles. Hence, we have shown that $\overleftrightarrow{MN} \perp \overleftrightarrow{AB}$. From this discussion follows the truth of:

Theorem 11. **If two lines intersect forming two congruent adjacent angles, the lines are perpendicular.**

The Meaning of "Equidistant"

In Fig. 3–23, if $PX = PY$, we say that the distance from P to X is equal to the distance from P to Y, or, more simply, P is *equidistant* from X and Y.

If M is also equidistant from X and Y ($MX = MY$), we say that P and M are each equidistant from X and Y.

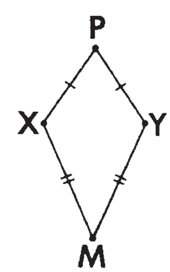


Fig. 3–23

Theorem 12. **Any point on the perpendicular bisector of a line segment is equidistant from the ends of the line segment.**

In Fig. 3–24, if $\overleftrightarrow{CD}$ is the perpendicular bisector of line segment $\overline{AB}$ and P is any point on $\overleftrightarrow{CD}$, then $PA = PB$.

Theorem 13. **If a point is equidistant from the ends of a line segment, the point must lie on the perpendicular bisector of the line segment.**

In Fig. 3–25, if $PA = PB$, then P must lie on line $\overleftrightarrow{CD}$, the perpendicular bisector of line segment $\overline{AB}$.

Theorem 14. **If two points are each equidistant from the ends of a line segment, the points determine the perpendicular bisector of the line segment.**

In Fig. 3–26, if $PA = PB$ and $QA = QB$, then the line which P and Q determine, line $\overleftrightarrow{CD}$, is the perpendicular bisector of line segment $\overline{AB}$.

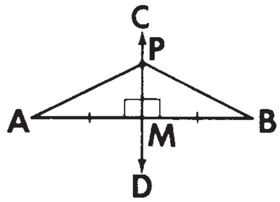


Fig. 3–24

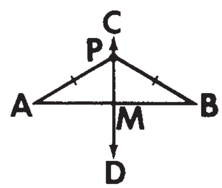


Fig. 3–25

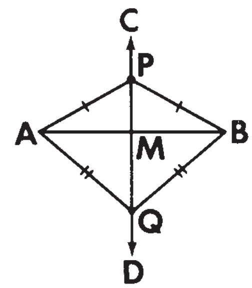


Fig. 3–26

Methods of Proving Lines or Line Segments Perpendicular

To prove that two intersecting lines or line segments are perpendicular, prove that one of the following statements is true:

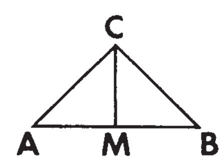
1. When the two lines or line segments intersect, they form right angles.
2. When the two lines or line segments intersect, they form congruent adjacent angles.
3. There are two points on one line or line segment, each of which is equidistant from the ends of the other line segment.

MODEL PROBLEM

Prove that the median to the base of an isosceles triangle is perpendicular to the base.

Given: $\triangle ABC$ with $\overline{CA} \cong \overline{CB}$.
$\overline{CM}$ is the median to base $\overline{AB}$.

To prove: $\overline{CM} \perp \overline{AB}$.



Method 1

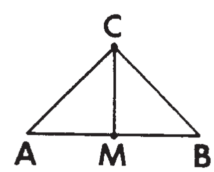
Plan: In order to prove $\overline{CM} \perp \overline{AB}$, prove that the adjacent angles which the segments form when they intersect, $\angle CMA$ and $\angle CMB$, are congruent. To prove $\angle CMA \cong \angle CMB$, prove that $\triangle CMA \cong \triangle CMB$ by s.s.s. $\cong$ s.s.s.

[The proof is given on the next page.]

Proof: *Statements*	*Reasons*
1. $\overline{CA} \cong \overline{CB}$. (s. $\cong$ s.)	1. Given.
2. $\overline{CM}$ is the median to base $\overline{AB}$.	2. Given.
3. $\overline{AM} \cong \overline{BM}$. (s. $\cong$ s.)	3. A median in a triangle divides the side to which it is drawn into two congruent parts.
4. $\overline{CM} \cong \overline{CM}$. (s. $\cong$ s.)	4. Reflexive property of congruence.
5. $\triangle CMA \cong \triangle CMB$.	5. s.s.s. $\cong$ s.s.s.
6. $\angle CMA \cong \angle CMB$.	6. Corresponding parts of congruent triangles are congruent.
7. $\angle CMA$ and $\angle CMB$ are adjacent angles.	7. Two angles are adjacent angles if they have a common vertex and a common side but do not have any interior points in common.
8. $\overline{CM} \perp \overline{AB}$.	8. Two line segments are perpendicular if they intersect and form congruent adjacent angles.

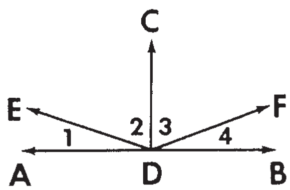
Method 2

Plan: In order to prove that $\overline{CM} \perp \overline{AB}$, prove that points C and M are each equidistant from the ends of line segment $\overline{AB}$.

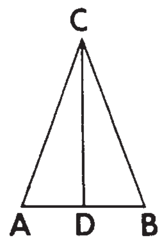


Proof: *Statements*	*Reasons*
1. $\overline{CA} \cong \overline{CB}$, or C is equidistant from A and B.	1. Given.
2. $\overline{CM}$ is a median.	2. Given.
3. $\overline{MA} \cong \overline{MB}$, or M is equidistant from A and B.	3. A median in a triangle divides the side to which it is drawn into two congruent parts.
4. $\overline{CM} \perp \overline{AB}$.	4. Two points each equidistant from the ends of a line segment determine the perpendicular bisector of the line segment.

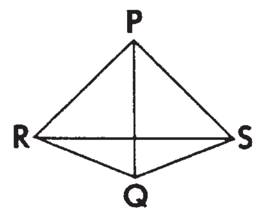
EXERCISES



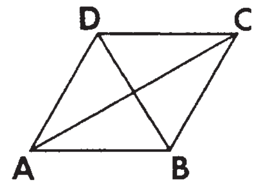
Ex. 1



Ex. 2

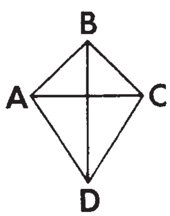


Ex. 3

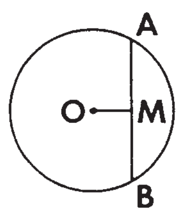


Ex. 4

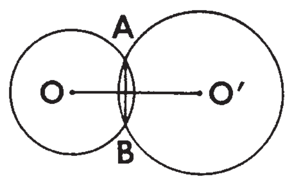
1. If $\angle 1 \cong \angle 4$ and $\angle 2 \cong \angle 3$, prove that $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$.
2. If $\overline{AC} \cong \overline{BC}$ and $\overline{CD}$ bisects $\angle ACB$, prove that $\overline{CD} \perp \overline{AB}$.
3. If $\overline{PR} \cong \overline{PS}$, and $\overline{QR} \cong \overline{QS}$, prove that $\overline{PQ} \perp \overline{RS}$.
4. If polygon $ABCD$ is equilateral, prove that $\overline{DB} \perp \overline{AC}$.



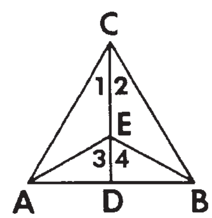
Ex. 5



Ex. 6



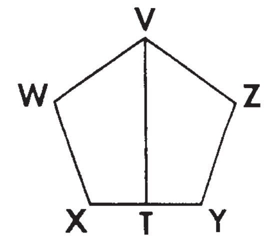
Ex. 7



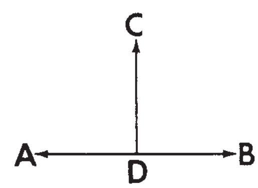
Ex. 8

5. *Given:* Polygon $ABCD$ in which $\overline{BD}$ bisects $\angle ABC$ and $\overline{DB}$ bisects $\angle ADC$.
   Prove: $\overline{BD}$ is the perpendicular bisector of $\overline{AC}$.
6. *Given:* In circle O, M is the midpoint of chord $\overline{AB}$.
   Prove: $\overline{OM} \perp \overline{AB}$. [*Hint:* Draw auxiliary segments $\overline{OA}$ and $\overline{OB}$.]
7. *Given:* Intersecting circles O and O'.
   Prove: $\overline{OO'}$ is the perpendicular bisector of $\overline{AB}$.
8. *Given:* In $\triangle ABC$, $\overleftrightarrow{CD}$ is a straight line, $\angle 1 \cong \angle 2$, and $\angle 3 \cong \angle 4$.
   Prove: $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$.

9. *Prove:* The bisector of the vertex angle of an isosceles triangle is perpendicular to the base of the triangle.
10. *Prove:* The line determined by the vertices of two isosceles triangles having a common base is perpendicular to this common base.



Ex. 11



Ex. 12–14

11. Polygon $VWXYZ$ is equiangular and equilateral. If $\overline{VT}$ bisects $\overline{XY}$, prove that $\overline{VT}$ is perpendicular to $\overline{XY}$.

12. $\overleftrightarrow{AB}$ is a straight line, $m\angle ADC = 3x + 18$, and $m\angle CDB = 4x - 6$. (*a*) Find the value of x. (*b*) Show that $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$.

13. If $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$, $m\angle ADC = 3x - y$, and $m\angle CDB = 2x + y$, find the value of x and the value of y.

14. If $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$, $m\angle CDA = 7x + y$, and $m\angle CDB = x + 4y$, find the value of x and the value of y.

15. At B, a point on line $\overleftrightarrow{CD}$, $\overleftrightarrow{BA}$ is drawn perpendicular to $\overleftrightarrow{CD}$. If $m\angle CBA = 7x + y$ and $m\angle DBA = 11x - y$, find x and y.

16. $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$ at E. $m\angle AEC = 3x$ and $m\angle AED = 5x - 60$. (*a*) Solve for x. (*b*) Show that $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{CD}$.

17. In triangle ABC, a line drawn from vertex A intersects $\overline{BC}$ in D. If $m\angle ADB = \frac{3}{2}x + 30$ and $m\angle ADC = 4x - 70$, show that $\overline{AD}$ is perpendicular to $\overline{BC}$.

18. In triangle RST, a line drawn from vertex R intersects $\overline{ST}$ in B. If $m\angle SBR = \frac{5}{2}x + 45$ and $m\angle TBR = 7x - 36$, show that $\overline{RB}$ is an altitude in triangle RST.

12. Method of Indirect Proof—Method of Elimination

In daily life, we often use a method of indirect proof, or indirect reasoning, which is illustrated by the following example:

Bill Tracy was charged with having committed a holdup in Boston on July 4, 1972 at 12 noon. At the trial, Bill's attorney, Mr. William Sawyer, said to the jury: "Bill Tracy cannot be guilty of this crime as charged because, according to the indictment, the crime was committed in Boston on July 4, 1972 at 12 noon. It is a matter of public record, reported in the newspapers, that Bill Tracy was the speaker at an Independence Day Rally that was held in New York City on July 4, 1972 at 12 noon. Your verdict in this case will have to be not guilty!"

In proving that "Bill Tracy is not guilty," the attorney, Mr. Sawyer, employed two principles of logic which we will now consider and accept without proof.

First, Mr. Sawyer showed that the proposition "Bill Tracy is guilty" led to the contradiction of the true proposition, "Bill Tracy was in New York City on July 4, 1972 at 12 noon." Therefore, he argued that the proposition "Bill Tracy is guilty" must be false. He made use of the following principle of logic, which we will now postulate.

Postulate 31. **If a proposition contradicts a true proposition, then it is false. (Postulate of Contradiction.)**

Second, in this case, there were only two possibilities: (1) "Bill Tracy is guilty." (2) "Bill Tracy is not guilty." One of these had to be true, the other false. Since Mr. Sawyer showed that the proposition "Bill Tracy is guilty" is false, he argued that the proposition "Bill Tracy is not guilty" must be true. He made use of the following principle of logic, which we will now postulate.

Postulate 32. **If one of a given set of propositions must be true, and all but one of those propositions have been proved to be false, then this one remaining proposition must be true. (Postulate of Elimination.)**

In geometry, we also use this method of indirect proof, called "the *method of elimination,*" in situations where it is difficult or impossible to use the method of direct proof which we have been using up to this point. In order to prove a conclusion by this method of indirect proof, we employ the following procedure:

1. List the conclusion and all other possibilities.
2. Prove all the other possibilities false. To do this, we usually use the "Postulate of Contradiction." We assume that each of these other possibilities is true and then show that the assumption leads to a contradiction of the *given*, a *postulate*, a *definition*, or a *previously proved theorem*.
3. State that the conclusion, which is the only remaining possibility, is true, making use of the "Postulate of Elimination."

KEEP IN MIND

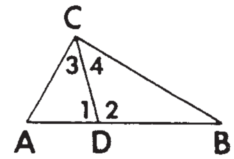
When using the method of elimination in indirect reasoning, be certain to list *all* the possibilities. One of these possibilities must be true.

MODEL PROBLEM

Prove that in a scalene triangle, the bisector of an angle cannot be perpendicular to the opposite side.

Given: ABC is a scalene triangle.
$\overline{CD}$ bisects $\angle ACB$.

To prove: $\overline{CD}$ is not $\perp$ $\overline{AB}$.



Plan: Use the method of indirect proof. List the possibilities: $\overline{CD} \perp \overline{AB}$ or $\overline{CD}$ is not $\perp$ $\overline{AB}$. Show that the possibility $\overline{CD} \perp \overline{AB}$ leads to a contradiction of the *given*.

Proof:

Statements	*Reasons*
1. Either $\overline{CD} \perp \overline{AB}$ or $\overline{CD}$ is not $\perp$ $\overline{AB}$.	1. There are only these two possibilities.
2. Suppose $\overline{CD} \perp \overline{AB}$.	2. One of the two possibilities.
3. $\angle 1 \cong \angle 2$. (a. $\cong$ a.)	3. When perpendicular line segments intersect, they form congruent adjacent angles.
4. $\overline{CD}$ bisects $\angle ACB$.	4. Given.
5. $\angle 3 \cong \angle 4$. (a. $\cong$ a.)	5. A bisector divides an angle into two congruent angles.
6. $\overline{CD} \cong \overline{CD}$. (s. $\cong$ s.)	6. Reflexive property of congruence.
7. $\triangle ACD \cong \triangle BCD$.	7. a.s.a. $\cong$ a.s.a.
8. $\overline{AC} \cong \overline{BC}$.	8. Corresponding sides of congruent triangles are congruent.
9. This contradicts the fact $\overline{AC}$ is not $\cong$ $\overline{BC}$.	9. In a scalene triangle, no two sides are congruent.
10. Hence, the supposition that $\overline{CD} \perp \overline{AB}$ is false.	10. Postulate of Contradiction.
11. $\overline{CD}$ is not $\perp$ $\overline{AB}$.	11. Postulate of Elimination.

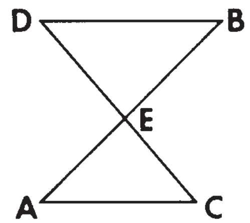
EXERCISES

Indirect Proof in Geometric Situations

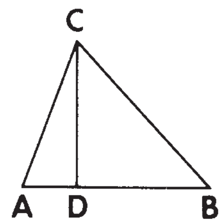
In each of the following, use the method of indirect proof:

1. *Prove:* If $m\angle A = 50$ and $m\angle B = 70$, then $\angle A$ and $\angle B$ are not complementary.

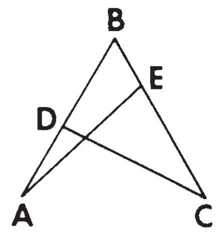
2. *Prove:* If $m\angle R = 130$ and $m\angle S = 80$, then $\angle R$ and $\angle S$ are not supplementary.



Ex. 3



Ex. 4



Ex. 5

3. *Given:* $\overline{DB}$ is not congruent to $\overline{AC}$, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are straight lines.
Prove: $\overline{AB}$ and $\overline{CD}$ do not bisect each other.
4. *Given:* $\overline{CD} \perp \overline{AB}$, $\overline{CA}$ is not congruent to $\overline{CB}$.
Prove: $\overline{AD}$ is not congruent to $\overline{DB}$.
5. *Given:* $\overline{AB} \cong \overline{BC}$, $\overline{BD}$ is not congruent to $\overline{BE}$.
Prove: $\angle A$ is not congruent to $\angle C$.

6. *Prove:* In a scalene triangle, the median to a side of the triangle cannot be perpendicular to that side.
7. *Prove:* In a scalene acute triangle, an altitude drawn to a side cannot bisect the angle which is opposite this side.
8. *Prove:* If a point is not equidistant from the ends of a line segment, the point does not lie on the perpendicular bisector of the line segment.
9. *Prove:* In a scalene triangle, no two angles are congruent.

Indirect Reasoning in Life Situations

10. One night Mr. Benson brought home a new table lamp which he tested when he bought it. He plugged the lamp into an outlet and found that it did not light. How could Mr. Benson prove by indirect reasoning that the outlet, and not the lamp, was defective?
11. A submarine sank off the coast of England. The hull was located on the ocean floor. Rescue parties worked frantically for several days in an effort to rescue survivors. At the end of this period of time, it was announced that all hands on board must be dead. Prove this conclusion by indirect reasoning.
12. Mrs. Moreno told her son Dan that his friend Kenny had broken his right leg in a football game the day before. The next day, when Dan went out for football practice, he found his friend Kenny practicing placement kicks with his right leg. Show by indirect reasoning how Dan knew that his mother had been mistaken.

13. Detective Collins was trying to prove that a suspect who pretended to be deaf was not deaf. He stood behind him and dropped a heavy book on the floor. The suspect was startled and jumped. Prove by indirect reasoning that the suspect was not deaf.
14. Ralph was told by his art teacher that his works were among the best in the class and were good enough to be exhibited. When an exhibition of the works of the class was held, Ralph was not invited to exhibit any of his paintings. Show by indirect reasoning how Ralph arrived at the conclusion that his teacher had only been encouraging him with flattery.
15. On June 27, 2002, Mr. Stone, in the company of his wife and several friends, attended a baseball game between the hours of 2 P.M. and 6 P.M. Several days later, Mr. Stone was arrested and charged with having run into another car on June 27, 2002 at 3 P.M., having caused the death of the driver and having left the scene of the accident. Mr. Stone's car, when picked up, had a badly smashed front right fender. How can Mr. Stone's lawyer prove to the jury, by indirect reasoning, that Mr. Stone was not guilty of the crime?
16. Jan, Linda, and Fran play in a dance orchestra. One plays the drums, one the clarinet, and one the piano. Jan and Linda hum the melody as they play their instruments. Jan brings her instrument with her to every engagement. What instrument does each girl play?
17. Give an illustration of indirect reasoning in a life situation.

13. True-False Exercises

If the statement is always true, write *true;* if the statement is not always true, write *false*.

1. Two triangles are congruent if they have two angles and the included side of one triangle congruent to the corresponding parts of the other.
2. The point of intersection of the bisectors of the base angles of an isosceles triangle and the ends of the base of the isosceles triangle are the vertices of another isosceles triangle.
3. If two angles of a triangle are congruent, the triangle is an isosceles triangle.
4. If two isosceles triangles have the same base, the line joining their vertices is perpendicular to the base.
5. If two angles of a polygon are congruent, the sides opposite these angles are congruent.
6. If the altitude drawn to the base of a triangle bisects the base, the triangle is an isosceles triangle.

7. Two triangles are congruent if three sides of one triangle are congruent respectively to three sides of the other.
8. If two adjacent angles are complementary, a side of one angle is perpendicular to a side of the other angle.
9. If two triangles have a pair of congruent sides and the altitudes drawn to these sides are also congruent, the triangles are congruent.
10. The bisectors of two supplementary angles are perpendicular to each other.

14. "Always, Sometimes, Never" Exercises

If the blank space in each of the following exercises is replaced by the word *always, sometimes,* or *never,* the resulting statement will be true. Select the word which will correctly complete each statement.

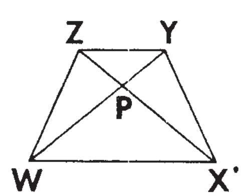
1. Two triangles are ______ congruent if two sides and the included angle of one are congruent respectively to two sides and the included angle of the other.
2. If the three sides of one triangle are congruent respectively to the three sides of another triangle, then the two triangles are ______ congruent.
3. If two angles are complementary, a side of one angle is ______ perpendicular to a side of the other angle.
4. An equilateral triangle is ______ congruent to a right triangle.
5. If two isosceles triangles have a leg and the base of one congruent to the corresponding parts of the other, the triangles are ______ congruent.
6. If the two legs of a right triangle are congruent to the corresponding legs of another right triangle, the triangles are ______ congruent.
7. Two triangles are ______ congruent if they have a pair of congruent sides and the altitudes drawn to these sides are congruent.
8. A leg of a right triangle is ______ one of the altitudes of the triangle.
9. If the three angles of a triangle are congruent, the triangle is ______ scalene.
10. In a triangle, two medians are ______ congruent.
11. The bisectors of two supplementary adjacent angles are ______ perpendicular to each other.
12. The median drawn to a side in an isosceles triangle is ______ perpendicular to that side.
13. The median to the base of an isosceles triangle ______ bisects the vertex angle.

14. The bisector of an angle of an isosceles triangle is ______ perpendicular to the opposite side.
15. A point on the perpendicular bisector of a line segment is ______ equidistant from the ends of the segment.
16. An equilateral triangle is ______ equiangular.
17. It is ______ possible to construct a right triangle if the given parts are the two legs of the triangles.
18. A median of a triangle ______ bisects the angle of the triangle through whose vertex it is drawn.
19. If two triangles are congruent, the corresponding angles are ______ congruent.
20. Two isosceles triangles are ______ congruent if they have congruent vertex angles.

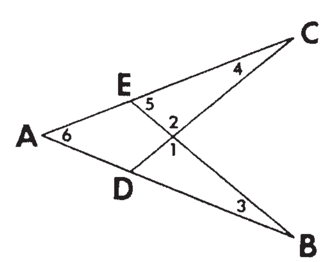
15. Multiple-Choice Exercises

In 1–5, write the letter preceding the word or expression that best completes the statement.

1. If in triangle ABC, $\overline{BD}$ is the median to side $\overline{AC}$, and $\triangle ABD \cong \triangle CBD$, then triangle ABC must be (*a*) scalene (*b*) isosceles (*c*) equilateral (*d*) right.
2. Two right triangles must be congruent if (*a*) the hypotenuse of one triangle is congruent to the hypotenuse of the other (*b*) an acute angle of one triangle is congruent to an acute angle of the other (*c*) two legs of one triangle are congruent to two legs of the other (*d*) the altitude drawn to the hypotenuse of one triangle is congruent to the altitude drawn to the hypotenuse of the other.
3. Two isosceles triangles are congruent if (*a*) the vertex angle of one triangle is congruent to the vertex angle of the other (*b*) a base angle of one triangle is congruent to a base angle of the other (*c*) a leg of one triangle is congruent to a leg of the other (*d*) a leg and the vertex angle of one triangle are congruent to a leg and the vertex angle of the other.



Ex. 4



Ex. 5

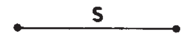
4. In the figure, it can be proved that $\overline{WY} \cong \overline{XY}$ if it is known that (*a*) $\angle YWX \cong \angle ZXW$ (*b*) $\angle YWZ \cong \angle YXZ$ (*c*) $\angle YXW \cong \angle PWX$ (*d*) $\overline{PW} \cong \overline{PX}$.
5. If $\overline{AB} \cong \overline{AC}$ in the figure, it can be proved that $\overline{CD} \cong \overline{BE}$ if it is also known that (*a*) $\angle 1 \cong \angle 2$ (*b*) $\angle 3 \cong \angle 4$ (*c*) $\angle 3 \cong \angle 5$ (*d*) $\angle 4 \cong \angle 6$.

16. Construction Exercises

The following exercises are to be done with straightedge and compasses. The basic constructions involved in these exercises appear in Chapter 13, which begins on page 614.

1. Construct a line segment equal in length to the sum of the lengths of two given line segments $\overline{AB}$ and $\overline{CD}$.
2. If l is the length of a given line segment $\overline{RS}$, construct a line segment whose length is (*a*) $2l$ (*b*) $4l$ (*c*) $1\frac{1}{2}l$ (*d*) $\frac{5}{4}l$.
3. Divide a given line segment $\overline{MN}$ into four congruent parts.
4. If x and y are the measures of two given line segments where $x > y$, construct a line segment whose measure is (*a*) $x + y$ (*b*) $2(x + y)$ (*c*) $\frac{1}{2}(x + y)$ (*d*) $x - y$ (*e*) $2(x - y)$ (*f*) $\frac{1}{2}(x - y)$.
5. Construct an angle congruent to a given angle B.
6. Construct the bisector of a given obtuse angle ABC.
7. In triangle ABC, construct the median to side $\overline{AC}$.
8. In obtuse triangle ABC with angle B the obtuse angle, construct the altitude to side $\overline{AB}$.
9. In a given acute triangle, construct (*a*) the three angle bisectors (*b*) the three medians (*c*) the three altitudes.
10. In a given obtuse triangle, construct (*a*) the three angle bisectors (*b*) the three medians (*c*) the three altitudes.
11. In a given right triangle, construct (*a*) the three angle bisectors (*b*) the three medians (*c*) the three altitudes.
12. Construct the complement of a given acute angle.
13. Construct the supplement of a given angle.
14. If r and s are the measures of two given angles where $r > s$, construct an angle whose measure is (*a*) $r + s$ (*b*) $2(r + s)$ (*c*) $\frac{1}{2}(r + s)$ (*d*) $r - s$ (*e*) $2(r - s)$ (*f*) $\frac{1}{2}(r - s)$.
15. At a point on a given line, construct a perpendicular to the given line.
16. From a point C outside line $\overleftrightarrow{AB}$, construct a line perpendicular to $\overleftrightarrow{AB}$.
17. Construct an angle which contains (*a*) $45°$ (*b*) $22\frac{1}{2}°$ (*c*) $135°$.

18. Referring to the figure, construct an equilateral triangle of side *s*.



Ex. 18

In 19–21, construct an isosceles triangle under the stated conditions.

19. Given a leg and the base.

20. Given a leg and the vertex angle.

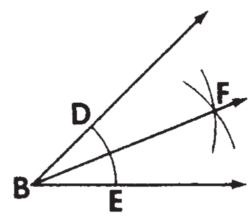
21. Given the base and a base angle.

22. Construct a right triangle given each of the two legs.

23. Construct an isosceles right triangle given one of the congruent legs.

24. Which one of the following statements is used in proving that $\overrightarrow{BF}$ bisects angle B in the figure?

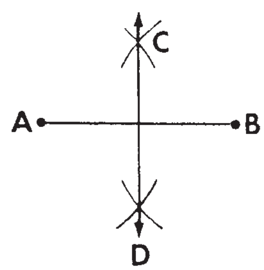
a. Two triangles are congruent if two sides and the included angle of one are congruent to the corresponding parts of the other.
b. An angle has one and only one bisector.
c. Two triangles are congruent if the three sides of one are congruent respectively to the three sides of the other.



Ex. 24

25. The figure shows the usual method of constructing the perpendicular bisector of a given line segment. Which statement, *a* or *b*, may be used to prove that $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$?

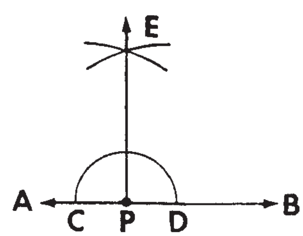
a. Two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.
b. All points on the perpendicular bisector of a line segment are equidistant from its endpoints.



Ex. 25

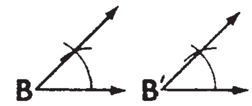
26. The figure shows the usual method of constructing a line perpendicular to a given line through a given point on the line. Which two of the following statements may be used to prove that $\overleftrightarrow{EP}$ is perpendicular to $\overleftrightarrow{AB}$?

a. When a straight angle is bisected, two right angles are formed.
b. Two right triangles are congruent if the legs of one triangle are congruent to the legs of the other triangle.
c. Two points, each equidistant from the ends of a line segment, determine the perpendicular bisector of the line segment.
d. Two triangles are congruent if two angles and the included side of one triangle are congruent respectively to two angles and the included side of the other triangle.



Ex. 26

27. The figure shows the usual method of constructing an angle congruent to a given angle. Which two of the following statements may be used to prove that angle B' is congruent to angle B?



Ex. 27

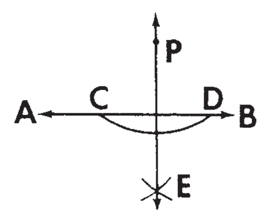
a. Two triangles are congruent if the three sides of one triangle are congruent respectively to the three sides of the other triangle.

b. Two triangles are congruent if two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of the other triangle.

c. Corresponding parts of congruent triangles are congruent.

d. Two triangles are congruent if two angles and the included side of one triangle are respectively congruent to two angles and the included side of the other triangle.

28. The figure shows the usual method of constructing a line perpendicular to a given line through a given point outside the line. Which two of the following statements may be used to prove that $\overleftrightarrow{PE}$ is perpendicular to $\overleftrightarrow{AB}$?



Ex. 28

a. From a given point outside a given line, one and only one perpendicular can be drawn to the line.

b. Two points each equidistant from the ends of a line segment determine the perpendicular bisector of the line segment.

c. The line which joins the vertices of two isosceles triangles on the same base bisects the common base at right angles.

d. Any point on the perpendicular bisector of a line segment is equidistant from the ends of the line segment.