CHAPTER XI

Coordinate Geometry

In ***coordinate geometry,*** or ***analytic geometry,*** algebraic principles and methods are used in studying geometric figures. This algebraic approach to geometry was first developed and systematized by the French mathematician René Descartes in the seventeenth century.

1. Plotting Points

Early in this book, we studied the real number line. We assumed that there exists a one-to-one correspondence between the set of points on a line and the set of real numbers; that is, each point on a line corresponds to a unique real number, and each real number corresponds to a unique point on a line.

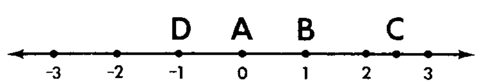


Fig. 11–1

In Fig. 11–1, point A, which is called the ***origin,*** is identified with the number 0; point B is identified with the number 1; point C corresponds to the number $2\frac{1}{2}$; point D corresponds to the number -1. The number that corresponds to a point on a line is called the ***coordinate*** of the point; the point to which a number corresponds is called the ***graph*** of the number. For example, the number 1 is the coordinate of point B, and point B is the graph of the number 1.

In view of the fact that in our study of geometry we deal not only with figures that are on a line, but also with figures that are in a plane, we will now devise a coordinate system in which every point in a plane is associated with a pair of numbers.

We will start with two number lines, called ***coordinate axes,*** which are perpendicular to each other. One line is horizontal and the other is vertical. The horizontal line is called the ***x-axis;*** the vertical line is called the ***y-axis.*** The point O at which the two axes intersect is called the ***origin.*** The axes divide the points of the plane which are not on the axes into four regions called ***quadrants.*** They are numbered in a counterclockwise direction as shown in Fig. 11–2.

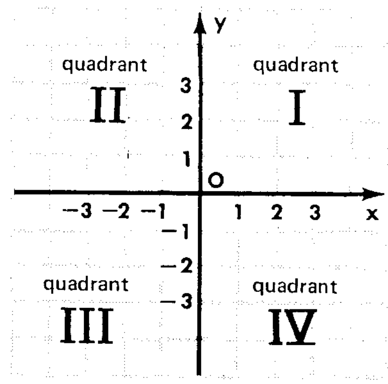


Fig. 11–2

The scales on the axes are marked so that the coordinates of points along the x-axis to the right of O are positive, and coordinates of points along the x-axis to the left of O are negative. Coordinates of points along the y-axis above O are positive, and coordinates of points along the y-axis below O are negative.

Let us use the two number lines to determine a pair of numbers that is associated with a point P in a plane. (See Fig. 11–3.) We draw, through P, a line that is perpendicular to the x-axis. This line intersects the x-axis at 2. The number 2 is called the *x-coordinate,* or ***abscissa,*** of point P. We also draw, through P, a line that is perpendicular to the y-axis. This line intersects the y-axis at 4. This number 4 is called the *y-coordinate,* or ***ordinate,*** of point P. The x-coordinate, 2, and the y-coordinate, 4, are called the *coordinates* of point P.

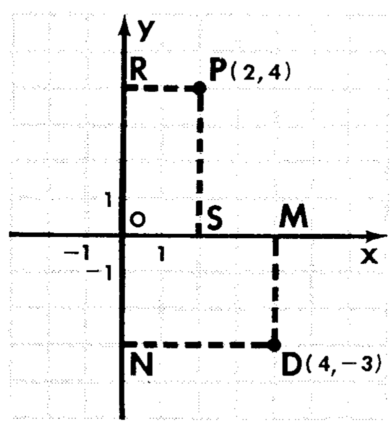


Fig. 11–3

To designate the coordinates of a point, the abscissa and ordinate are written in parentheses with a comma between them. Note that the abscissa is always written first. The pair of numbers (2, 4) is called the coordinates of point P. Since the abscissa was written first, the pair of numbers (2, 4) is called an ***ordered pair of numbers.*** The pair of numbers (2, 4) is not the same as the pair of numbers (4, 2) because (4, 2) locates a point whose abscissa is 4 and whose ordinate is 2.

Similarly, in the plane pictured in Fig. 11–3, the ordered pair of numbers $(4, -3)$ is associated with point D, whose abscissa is 4 and whose ordinate is -3. Point D is the graph of the ordered pair of numbers $(4, -3)$. When we graph the point described by an ordered pair of numbers, we are *plotting the point.*

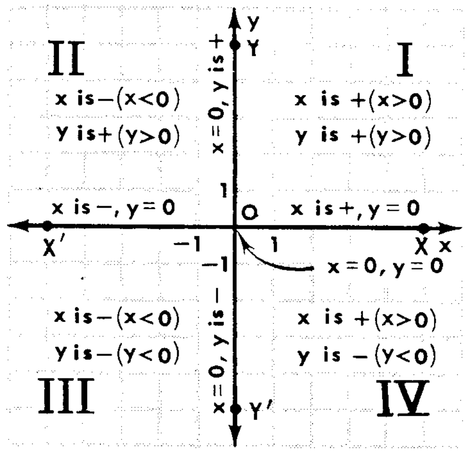


Fig. 11–4

In Fig. 11–4, *since* all points in quadrant I must be to the right of the y-axis and above the x-axis, every point in quadrant I represents an ordered pair of numbers whose abscissa and ordinate are both positive.

Since all points in quadrant II must be to the left of the y-axis and above the x-axis, every point in quadrant II represents an ordered pair of numbers whose abscissa is negative and whose ordinate is positive.

Since all points in quadrant III must be to the left of the y-axis and below the x-axis, every point in quadrant III represents an ordered pair of numbers whose abscissa and ordinate are both negative.

Since all points in quadrant IV must be to the right of the y-axis and below the x-axis, every point in quadrant IV represents an ordered pair of numbers whose abscissa is positive and whose ordinate is negative.

Every point on $\overrightarrow{OX}$, the positive part of the x-axis (except point O), represents an ordered pair of numbers whose abscissa is positive and whose ordinate is 0. Every point on $\overrightarrow{OX'}$, the negative part of the x-axis (except point O), represents an ordered pair of numbers whose abscissa is negative and whose ordinate is 0. Every point on $\overrightarrow{OY}$, the positive part of the y-axis (except point O), represents an ordered pair of numbers whose abscissa is 0 and whose ordinate is positive. Every point on $\overrightarrow{OY'}$, the negative part of the y-axis (except point O), represents an ordered pair of numbers whose abscissa is 0 and whose ordinate is negative.

The origin, point O, represents the ordered pair of numbers (0, 0); that is, the x-coordinate of the origin is 0, and the y-coordinate of the origin is 0.

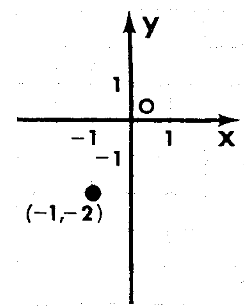
Note that the graph of every ordered pair of numbers (x, y) is a unique point in the plane. Also, every point in the plane represents a unique ordered pair of numbers (x, y). We say that there is a one-to-one correspondence between the sets of points in the plane and the set of ordered pairs of real numbers. This correspondence is called a *coordinate system*.

Since the axes in the coordinate system that we are using are perpendicular to each other, this system is called a *rectangular coordinate system*. It is also called the *Cartesian coordinate system*, named after René Descartes, its originator.

MODEL PROBLEM

Describe the location of the point $(-1, -2)$, and then plot the point on coordinate graph paper. [The solution is given on the next page.]

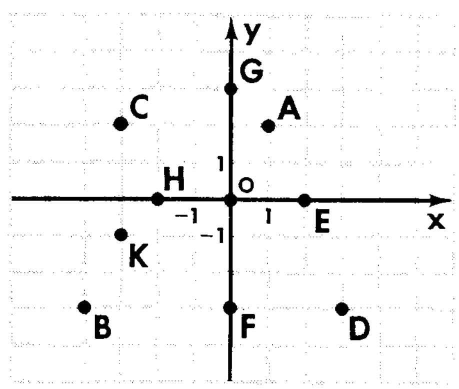
Solution: The ordered number pair describes a point in quadrant III which is 1 unit to the left of the y-axis and 2 units below the x-axis.



EXERCISES

For each point named in 1–10, refer to the graph below and tell:

a. the abscissa of the point.
b. the ordinate of the point.
c. the coordinates of the point given as an ordered pair.



Ex. 1–10

1. point A **2.** point B **3.** point C **4.** point D **5.** point E
6. point F **7.** point G **8.** point H **9.** point K **10.** point O

In 11–26, describe the location of the point, and then plot the point on coordinate graph paper.

11. (4, 5) **12.** (−2, 5) **13.** (3, −4) **14.** (−5, −4)
15. (2, 7) **16.** (−6, 3) **17.** $(5, -2\frac{1}{2})$ **18.** $(-\frac{3}{4}, -7)$
19. (4, 0) **20.** (−5, 0) **21.** (3, 0) **22.** (−8, 0)
23. (0, 6) **24.** (0, −8) **25.** (0, 1) **26.** (0, −9)

27. If a point is on the x-axis, what is the value of its ordinate?

28. What is the value of the abscissa of every point which is on the y-axis?

29. What are the coordinates of the origin?

30. Tell the type of number that the abscissa of a point and the ordinate of that point must be if the point lies in quadrant (*a*) I (*b*) II (*c*) III (*d*) IV.

31. If x is a positive number and y is a negative number, name the quadrant in which each of the following points lies:
a. (x, y) *b.* $(x, -y)$ *c.* $(-x, y)$ *d.* $(-x, -y)$

32. If x is a negative number and y is a negative number, name the quadrant in which each of the following points lies:
a. (x, y) *b.* $(x, -y)$ *c.* $(-x, y)$ *d.* $(-x, -y)$

2. Informal Proofs in Coordinate Geometry

We will now make use of several properties of graphs. We will illustrate and discuss these properties, but we will not prove them formally.

Notice the following properties in the graph in Fig. 11–5:

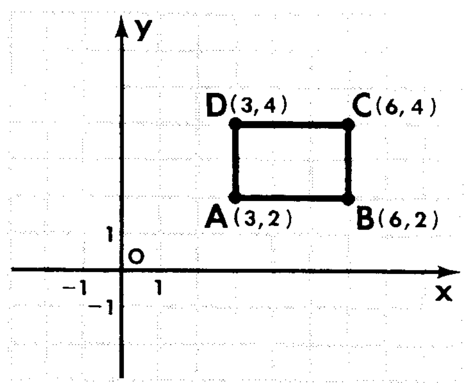


Fig. 11–5

1. Points A and D have the same abscissa, 3. Thus, $\overleftrightarrow{AD}$ is parallel to the y-axis. Similarly, $\overleftrightarrow{BC}$ is parallel to the y-axis.
2. Points A and B have the same ordinate, 2. Thus, $\overleftrightarrow{AB}$ is parallel to the x-axis. Similarly, $\overleftrightarrow{DC}$ is parallel to the x-axis.
3. $\overleftrightarrow{DC}$ and $\overleftrightarrow{AB}$ are parallel to each other because they are each parallel to the x-axis.
4. $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ are parallel to each other because they are each parallel to the y-axis.
5. $\overleftrightarrow{AD} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{BC} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{BC} \perp \overleftrightarrow{DC}$, and $\overleftrightarrow{AD} \perp \overleftrightarrow{DC}$ because on rectangular coordinate graph paper every horizontal line is perpendicular to every vertical line. Then $\angle DAB$, $\angle ABC$, $\angle BCD$, and $\angle CDA$ are right angles.
6. The length of $\overline{AB}$ is 3 units. We can find this length either by counting the number of units contained in line segment $\overline{AB}$ or by subtracting 3 from 6. Similarly, the length of $\overline{DC}$ is 3 units. Therefore, $\overline{AB} \cong \overline{DC}$.
7. The length of $\overline{AD}$ is 2 units. We can find this length either by counting the number of units contained in line segment $\overline{AD}$ or by subtracting 2 from 4. Similarly, the length of $\overline{BC}$ is 2 units. Therefore, $\overline{AD} \cong \overline{BC}$.

We will use the preceding properties of graphs in developing informal proofs for exercises in coordinate geometry. These proofs may be written in paragraph form or they may be arranged step-by-step. A reason should be given only when the reason is not one of the properties of graphs discussed in this unit.

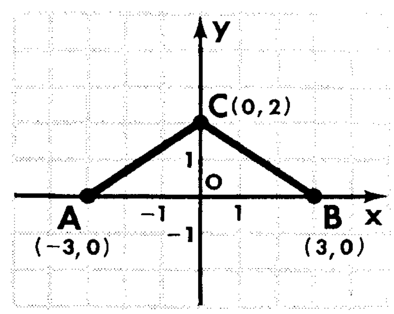
MODEL PROBLEM

Given: The vertices of $\triangle ABC$ are $A(-3, 0)$, $B(3, 0)$, and $C(0, 2)$.

To show: $\triangle ABC$ is an isosceles triangle.

Plan: Use congruent triangles to prove $\overline{CA} \cong \overline{CB}$.

Solution 1 (step by step):



1. Since the y-axis $\perp$ the x-axis, rt.$\angle COA \cong$ rt.$\angle COB$.
2. $OA = 3$ units and $OB = 3$ units. Therefore, $\overline{OA} \cong \overline{OB}$.
3. $\overline{OC} \cong \overline{OC}$ by the reflexive property of congruence.
4. $\triangle COA \cong \triangle COB$ by s.a.s. $\cong$ s.a.s.
5. Therefore, $\overline{CA} \cong \overline{CB}$, and $\triangle ABC$ is an isosceles triangle.

Solution 2 (paragraph form):

$\overleftrightarrow{OC} \perp \overleftrightarrow{AB}$, making $\angle COA$ and $\angle COB$ congruent right angles. Since O is $(0, 0)$ and lies halfway between $(-3, 0)$ and $(3, 0)$, it is the midpoint of $\overline{AB}$, and $\overline{OA} \cong \overline{OB}$. $\overline{OC} \cong \overline{OC}$ by the reflexive property of equality. $\triangle COA \cong \triangle COB$ by s.a.s. $\cong$ s.a.s. Therefore, $\overline{CA} \cong \overline{CB}$, and $\triangle ABC$ is isosceles because it is a triangle which has two congruent sides.

EXERCISES

1. Plot the following points: $A(0, 0)$, $B(0, 3)$, $C(4, 0)$. Connect these points in the order given. What kind of triangle is triangle ABC? Show why your conclusion is correct.
2. Show that the triangle whose vertices are the points $(-1, 1)$, $(-3, 2)$, and $(-1, 2)$ is a right triangle.
3. Plot the following points: $A(-2, 0)$, $B(0, -3)$, $C(2, 0)$. Connect the points in the order given. What kind of triangle is triangle ABC? Give an informal proof of your conclusion.

4. Show that the triangle whose vertices are the points (1, 1), (9, 4), (1, 7) is an isosceles triangle.
5. Plot the following points: $A(5, 3)$, $B(-5, 3)$, $C(-5, -3)$, $D(5, -3)$. Connect these points in the order given. What kind of quadrilateral is $ABCD$? Prove your conclusion informally.
6. Draw the quadrilateral which has the following points as its vertices. What type of quadrilateral is formed? Prove your conclusion informally.
   a. (1, 1), (4, 1), (4, −2), (1, −2) *b.* (2, 2), (3, 4), (5, 2), (6, 4)
   c. (−1, −1), (−1, −3), (3, −1), (3, −3) *d.* (6, 0), (9, 4), (6, 8), (3, 4)
7. Draw the quadrilateral which has the following points as its vertices. What type of quadrilateral appears to be formed?
   a. (2, 2), (3, 4), (5, 4), (9, 2) *b.* (0, 0), (−6, 0), (−4, 2), (−2, 2)
   c. (1, 2), (3, 5), (5, 2), (3, −1) *d.* (−3, 1), (−1, 3), (1, 3), (3, 1)
8. Plot the following points: $A(0, 0)$, $B(4, -2)$, $C(8, 0)$, $D(4, 2)$. (*a*) Connect these points in the given order. (*b*) Show that $\overleftrightarrow{BD}$ is the perpendicular bisector of $\overline{AC}$. (*c*) What type of quadrilateral is $ABCD$?
9. Show that the quadrilateral whose vertices are the points (−1, 1), (1, −2), (3, 1), and (1, 4) is a rhombus.
10. Triangle ABC has as its vertices the points $A(1, 1)$, $B(4, 1)$, and $C(4, 5)$. Triangle DEF has as its vertices the points $D(1, -1)$, $E(4, -1)$, and $F(4, -5)$. Show that triangle ABC is congruent to triangle DEF.

3. The Distance Between Two Points

Finding the Distance Between Two Points Having the Same Ordinate

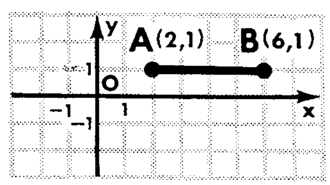


Fig. 11–6

Since the points $A(2, 1)$ and $B(6, 1)$ in Fig. 11–6 have the same ordinate, 1, they lie on a line which is parallel to the x-axis. We can discover that the distance between point A and point B is 4, either by counting the number of units contained in $\overline{AB}$, or by finding the difference between the abscissa of B, 6, and the abscissa of A, 2, obtaining $6 - 2 = 4$.

Suppose we had subtracted the abscissa of B, 6, from the abscissa of A, 2. We would then have obtained the result $2 - 6$, or -4, which is different from the previous result, 4. We can resolve this inconsistency by agreeing that the distance between A and B is equal to the absolute value of the difference of their abscissas. Then, the order in which we subtract the abscissas would not matter because

$$|6 - 2| = |4| = 4 \qquad \text{also} \qquad |2 - 6| = |-4| = 4$$

Postulate 58. **The distance between two points having the same ordinate is the absolute value of the difference of their abscissas.**

In Fig. 11–7, the distance, d, between two points $A(x_1, y_1)$ and $B(x_2, y_1)$ having the same ordinate, y_1, is the absolute value of the result obtained when x_1, the abscissa of point A, is subtracted from x_2, the abscissa of point B, or

$$d = |x_2 - x_1|$$

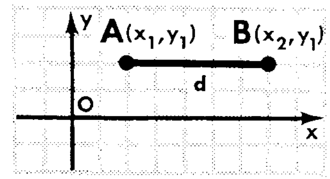


Fig. 11–7

Finding the Distance Between Two Points Having the Same Abscissa

Since the points $E(3, 2)$ and $F(3, 5)$ in Fig. 11–8 have the same abscissa, 3, they lie on a line which is parallel to the y-axis. We can discover that the distance between point E and point F is 3, either by counting the number of units contained in $\overline{EF}$, or by finding the difference between the ordinate of F, 5, and the ordinate of E, 2, or $5 - 2 = 3$.

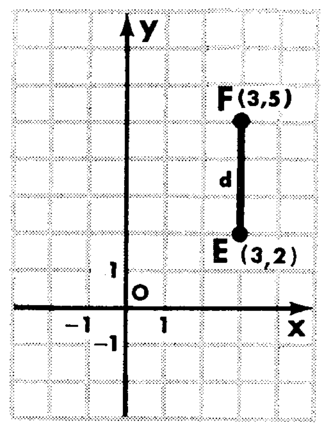


Fig. 11–8

Suppose we had subtracted the ordinate of F, 5, from the ordinate of E, 2. We would then have obtained the result $2 - 5$, or -3, which is different from the previous result, 3. We can resolve this inconsistency by agreeing that the distance between E and F is equal to the absolute value of the difference of their ordinates. Then, the order in which we subtract the ordinates would not matter because

$$|5 - 2| = |3| = 3 \qquad \text{also} \qquad |2 - 5| = |-3| = 3$$

Postulate 59. **The distance between two points having the same abscissa is the absolute value of the difference of their ordinates.**

In Fig. 11–9, the distance, d, between two points $E(x_1, y_1)$ and $F(x_1, y_2)$ having the same abscissa, x_1, is the absolute value of the result obtained when y_1, the ordinate of point E, is subtracted from y_2, the ordinate of point F, or

$$d = |y_2 - y_1|$$

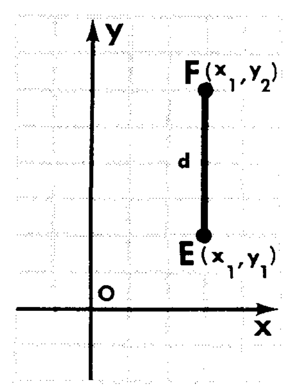


Fig. 11–9

Finding the Distance Between Any Two Points

The methods previously discussed cannot be used to find the distance between the point $A(2, 3)$ and the point $B(5, 7)$ because these two points have different abscissas and different ordinates. However, if we form a right triangle as shown in Fig. 11–10, we can use the Pythagorean Theorem to find d, the length of $\overline{AB}$.

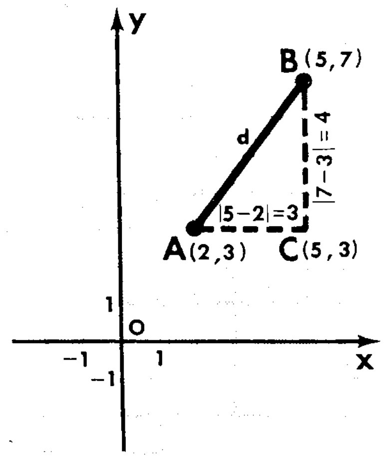


Fig. 11–10

The coordinates of C, the vertex of right angle ACB, are $(5, 3)$.

The length of $\overline{AC} = |5-2| = |3| = 3$, and the length of $\overline{BC} = |7-3| = |4| = 4$. Therefore,

$$d^2 = 3^2 + 4^2$$
$$d^2 = 9 + 16$$
$$d^2 = 25$$
$$d = 5$$

Let us now derive a formula for the distance, d, between any two points $R(x_1, y_1)$ and $S(x_2, y_2)$ in Fig. 11–11.

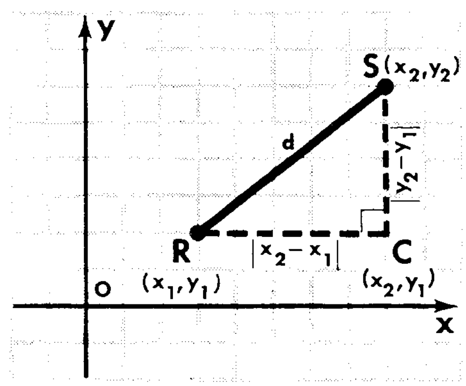


Fig. 11–11

1. Form right triangle RCS by drawing through S a line parallel to the y-axis and by drawing through R a line parallel to the x-axis, with the two lines intersecting at C.
2. Since the coordinates of point C are (x_2, y_1), then $RC = |x_2 - x_1|$ and $CS = |y_2 - y_1|$.
3. In right triangle RCS, let $RS = d$.
4. $(RS)^2 = (RC)^2 + (CS)^2$, or
   $d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$, or
   $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
   $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

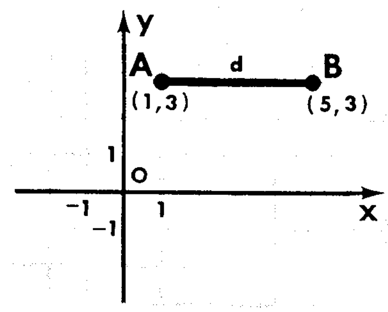
Theorem 147. **The distance, d, between two points (x_1, y_1) and (x_2, y_2) is given by the formula**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

MODEL PROBLEMS

1. Find the distance between the point $A(1, 3)$ and the point $B(5, 3)$.

Solution:



1. Let the coordinates of point A be (x_1, y_1). Then $x_1 = 1$, $y_1 = 3$.
2. Let the coordinates of point B be (x_2, y_2). Then $x_2 = 5$, $y_2 = 3$.
3. Since points A and B have the same ordinate, 3, the distance, d, between point A and point B is given by the formula

$$d = |x_2 - x_1| = |5 - 1| = |4| = 4$$

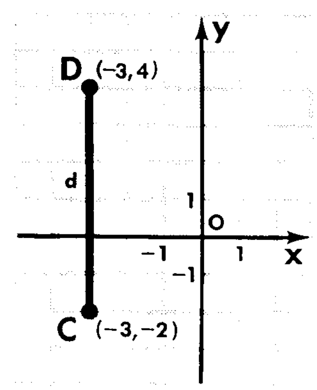
Answer: 4.

NOTE. It is also possible to use the general formula for the distance between two points.

1. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. $d = \sqrt{(5 - 1)^2 + (3 - 3)^2} = \sqrt{(4)^2 + (0)^2} = \sqrt{16 + 0} = \sqrt{16} = 4$

2. Find the distance between the point $C(-3, -2)$ and the point $D(-3, 4)$.

Solution:



1. Let the coordinates of point C be (x_1, y_1). Then $x_1 = -3$, $y_1 = -2$.
2. Let the coordinates of point D be (x_2, y_2). Then $x_2 = -3$, $y_2 = 4$.
3. Since points C and D have the same abscissa, -3, the distance, d, between point C and point D is given by the formula

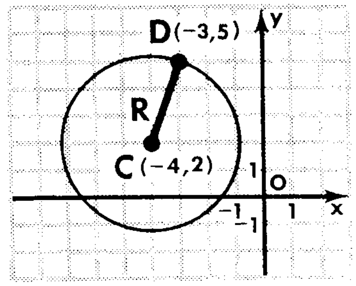
$$d = |y_2 - y_1|$$
$$d = |4 - (-2)| = |4 + 2| = |6| = 6$$

Answer: 6.

NOTE. It is also possible to use the general formula for the distance between two points.

1. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. $d = \sqrt{[-3 - (-3)]^2 + [4 - (-2)]^2} = \sqrt{(-3 + 3)^2 + (4 + 2)^2}$
3. $d = \sqrt{(0)^2 + (6)^2} = \sqrt{0 + 36} = \sqrt{36} = 6$

3. A circle whose center is at $C(-4, 2)$ passes through the point $D(-3, 5)$. Find R, the length of radius $\overline{CD}$, in radical form.



Solution:

1. Let the coordinates of point C be (x_1, y_1). Then $x_1 = -4$, $y_1 = 2$.
2. Let the coordinates of point D be (x_2, y_2). Then $x_2 = -3$, $y_2 = 5$.
3. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
4. $R = \sqrt{[-3 - (-4)]^2 + [5 - 2]^2} = \sqrt{(-3 + 4)^2 + (5 - 2)^2}$
5. $R = \sqrt{(1)^2 + (3)^2} = \sqrt{1 + 9} = \sqrt{10}$

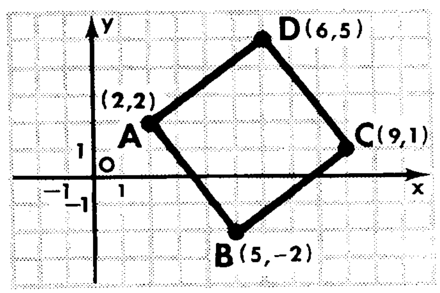
Answer: $R = \sqrt{10}$.

Note. When we use the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ to find the distance between two points, either of the points may be named (x_1, y_1) provided that the other point is named (x_2, y_2). This is true because $(x_2 - x_1)^2 = (x_1 - x_2)^2$ and $(y_2 - y_1)^2 = (y_1 - y_2)^2$.

4. *Given:* The quadrilateral with vertices $A(2, 2)$, $B(5, -2)$, $C(9, 1)$, and $D(6, 5)$.

To show: $ABCD$ is a rhombus.

Plan: To show that $ABCD$ is a rhombus, show that it is a quadrilateral with four sides of equal lengths.



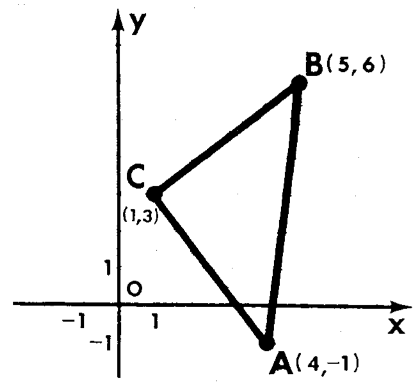
Solution:

1. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. $AB = \sqrt{(5 - 2)^2 + (-2 - 2)^2} = \sqrt{(3)^2 + (-4)^2}$
3. $AB = \sqrt{9 + 16} = \sqrt{25} = 5$
4. $BC = \sqrt{[9 - 5]^2 + [1 - (-2)]^2} = \sqrt{(4)^2 + (1 + 2)^2}$
5. $BC = \sqrt{(4)^2 + (3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$
6. $CD = \sqrt{(9 - 6)^2 + (1 - 5)^2} = \sqrt{(3)^2 + (-4)^2}$
7. $CD = \sqrt{9 + 16} = \sqrt{25} = 5$
8. $DA = \sqrt{(6 - 2)^2 + (5 - 2)^2} = \sqrt{(4)^2 + (3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$
9. $AB = BC = CD = DA$.
10. Since $ABCD$ is an equilateral quadrilateral, it is a rhombus.

5. *Given:* The triangle whose vertices are $A(4, -1)$, $B(5, 6)$, and $C(1, 3)$.

To show: $\triangle ABC$ is an isosceles right triangle.

Plan: To show that $\triangle ABC$ is an isosceles right triangle, show that two sides are equal in length and that the square of the length of the longest side is equal to the sum of the squares of the lengths of the other two sides.



Solution:

1. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. $AB = \sqrt{[5-4]^2 + [6-(-1)]^2} = \sqrt{(5-4)^2 + (6+1)^2}$
3. $AB = \sqrt{(1)^2 + (7)^2} = \sqrt{1+49} = \sqrt{50}$
4. $BC = \sqrt{(5-1)^2 + (6-3)^2} = \sqrt{(4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$
5. $CA = \sqrt{(4-1)^2 + (-1-3)^2} = \sqrt{(3)^2 + (-4)^2}$
6. $CA = \sqrt{9+16} = \sqrt{25} = 5$
7. $BC = CA$. Therefore, triangle ABC is an isosceles triangle.
8. $(AB)^2 = (\sqrt{50})^2 = 50$, $(BC)^2 = (5)^2 = 25$, $(CA)^2 = (5)^2 = 25$.
9. Since $50 = 25 + 25$, $(AB)^2 = (BC)^2 + (CA)^2$, and triangle ABC is a right triangle.
10. Therefore, triangle ABC is an isosceles right triangle.

EXERCISES

1. Find the distance between each of the following pairs of points: [Answers may be left in radical form.]

a. (2, 7) and (12, 7) *b.* (−3, 4) and (5, 4) *c.* (−1, −2) and (−1, 4)
d. (0, 0) and (4, 3) *e.* (0, 0) and (−3, 4) *f.* (−6, −8) and (0, 0)
g. (1, 4) and (4, 8) *h.* (4, 2) and (−2, 10) *i.* (−10, −12) and (5, 8)

j. (6, 4) and (3, 6) *k.* (−5, 0) and (−9, 6) *l.* (0, 0) and (−2, 5)
m. (0, c) and (b, 0) *n.* (0, 0) and (a, b)

2. Find the length of the line segment joining the points whose coordinates are: [Answers may be left in radical form.]
a. (5, 2) and (8, 6) *b.* (−5, 1) and (7, 6)
c. (0, 5) and (−3, 3) *d.* (−4, −5) and (1, −2)

3. Find the lengths of the sides of a triangle whose vertices are: [Answers may be left in radical form.]
a. (0, 0), (8, 0), (4, 3) *b.* (1, 5), (5, 5), (5, 1)
c. (3, 6), (−1, 3), (5, −5) *d.* (6, −3), (0, 4), (8, −1)
e. (−1, 7), (0, 0), (8, 4) *f.* (−4, 2), (−1, 6), (5, 4)

4. Find the length of the shortest side of the triangle whose vertices are $R(-2, -1)$, $S(1, 3)$, $T(1, 10)$.

5. Show that points A, B, and C are collinear; that is, they lie on the same straight line. [*Hint:* Show that $AB + BC = AC$.]
a. $A(0, 0)$, $B(5, 12)$, $C(10, 24)$ *b.* $A(-1, -2)$, $B(2, 2)$, $C(8, 10)$
c. $A(-2, -2)$, $B(0, 2)$, $C(1, 4)$ *d.* $A(-1, 2)$, $B(2, -1)$, $C(4, -3)$

6. Show that the triangles which have the following vertices are isosceles triangles:
a. (2, 3), (5, 7), (1, 4) *b.* (1, 0), (5, 0), (3, 4)
c. (7, −1), (2, −2), (3, 3) *d.* (4, −7), (−3, −4), (7, 0)

7. Show that the triangles which have the following vertices are right triangles:
a. (1, 1), (4, 5), (4, 1) *b.* (5, 6), (8, 5), (2, −3)
c. (−1, 0), (6, 1), (2, 4) *d.* (−4, −1), (0, −5), (1, 4)

8. The vertices of triangle ABC are $A(2, 4)$, $B(5, 8)$, and $C(9, 5)$. (*a*) Show that triangle ABC is an isosceles triangle. (*b*) Show that triangle ABC is a right triangle.

9. Show that the line segments joining the points (−1, 3), (9, 3), and (4, 8) form an isosceles right triangle.

10. The points (1, 1), (7, 1), (7, 4), and (1, 4) are the vertices of a rectangle. Show that the diagonals are equal in length.

11. Show that the quadrilaterals which have the following vertices are parallelograms.
a. (1, 2), (2, 5), (5, 7), (4, 4) *b.* (−1, 1), (−3, 4), (1, 5), (3, 2)

12. Show that the quadrilaterals which have the following vertices are rhombuses:
a. (1, 1), (5, 3), (7, 7), (3, 5) *b.* (−3, 2), (−2, 6), (2, 7), (1, 3)

13. *a.* Show that the quadrilateral whose vertices are (1, 4), (4, 9), (−1, 12), (−4, 7) is equilateral.
b. Show that the diagonals of this quadrilateral are equal in length.
c. What type of quadrilateral have you shown this to be?

14. The vertices of trapezoid $ABCD$ are $A(1, -4)$, $B(10, -4)$, $C(9, 2)$, and $D(2, 2)$. (*a*) Show that $ABCD$ is an isosceles trapezoid. (*b*) Show that the length of diagonal $\overline{AC}$ equals the length of diagonal $\overline{BD}$.

15. Find the length of a radius of a circle whose center is at (0, 0) and which passes through the point (12, 5).

16. The point (2, 4) is on the circle whose center is (6, 1). Find the length of a radius of the circle.

17. Find the length of a radius of a circle whose center is at the origin and which passes through the point (−3, 4).

18. A circle whose center is at the point (5, 6) passes through the origin. Without constructing the circle, show that the point (11, 11) lies on the circle and that the point (9, 12) does not lie on the circle.

19. Show that a circle whose center is (2, 3) and which passes through the point (8, 11) also passes through the points (10, 9), (−4, −5), (8, −5).

20. A circle whose center is the point (2, 3) is tangent to the x-axis. Find the coordinates of the point of tangency.

21. A circle whose center is at the point (−6, 3) is tangent to the y-axis. Find the coordinates of the point of tangency.

22. (*a*) A circle whose center is at (6, 8) passes through the point (12, 16). Find the length of a radius of the circle. (*b*) On the circle given in *a*, there is another point whose abscissa is 12. Find its ordinate. (*c*) Find the distance of the center from the origin.

23. Point $A(-6, 2)$ is the center of a circle, and point $C(-3, 8)$ lies on the circle. A line segment joins point $B(9, 2)$ to point C. Using coordinate geometry, show that $\overleftrightarrow{BC}$ is tangent to the circle at point C.

4. The Midpoint of a Line Segment

Theorem 148. **Each coordinate of the midpoint of a line segment is equal to one-half the sum of the corresponding coordinates of the endpoints of the line segment.**

The coordinates of the midpoint $M(x_m, y_m)$ of line segment $\overline{AB}$ which joins any point $A(x_1, y_1)$ and any other point $B(x_2, y_2)$ are given by the formulas

$$x_m = \frac{1}{2}(x_1 + x_2) \qquad y_m = \frac{1}{2}(y_1 + y_2)$$

INFORMAL PROOF FOR $x_m = \frac{1}{2}(x_1 + x_2)$:

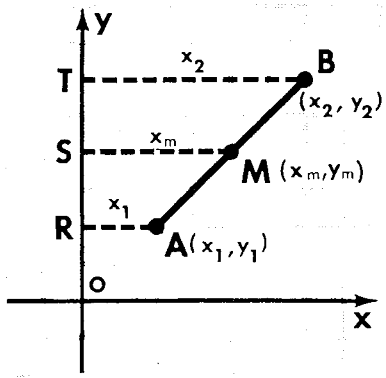


Fig. 11–12

In Fig. 11–12, draw $\overleftrightarrow{AR} \perp$ y-axis, $\overleftrightarrow{MS} \perp$ y-axis, and $\overleftrightarrow{BT} \perp$ y-axis. Then $\overleftrightarrow{RA} \parallel \overleftrightarrow{SM} \parallel \overleftrightarrow{TB}$. Since M is the midpoint of $\overline{AB}$, $AM = MB$. Therefore, $RS = ST$ because three parallel lines which cut off segments whose lengths are equal on one transversal cut off segments whose lengths are equal on any transversal. $\overline{SM}$ is a median in trapezoid $RABT$ and its length is equal to one-half the sum of the lengths of the bases $\overline{RA}$ and $\overline{TB}$. Therefore, $SM = \frac{1}{2}(RA + TB)$, or $x_m = \frac{1}{2}(x_1 + x_2)$.

INFORMAL PROOF FOR $y_m = \frac{1}{2}(y_1 + y_2)$:

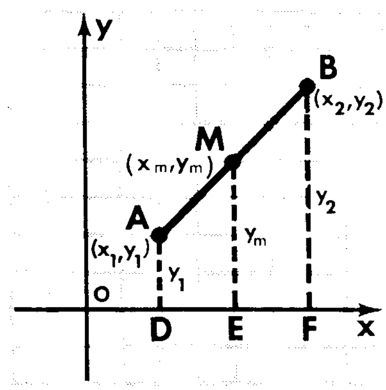


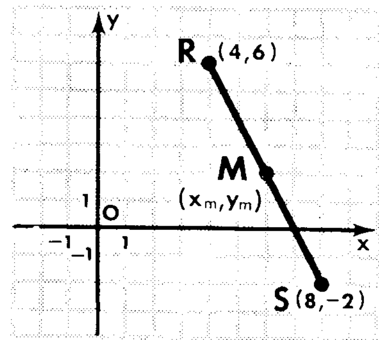
Fig. 11–13

In Fig. 11–13, draw $\overleftrightarrow{AD} \perp$ x-axis, $\overleftrightarrow{ME} \perp$ x-axis, and $\overleftrightarrow{BF} \perp$ x-axis. Then $\overleftrightarrow{DA} \parallel \overleftrightarrow{EM} \parallel \overleftrightarrow{FB}$. Since M is the midpoint of $\overline{AB}$, $AM = MB$. Therefore, $DE = EF$ because three parallel lines which cut off segments whose lengths are equal on one transversal cut off segments whose lengths are equal on any transversal. $\overline{EM}$ is a median in trapezoid $BFDA$ and its length is equal to one-half the sum of the lengths of the bases $\overline{DA}$ and $\overline{FB}$. Therefore, $EM = \frac{1}{2}(DA + FB)$, or $y_m = \frac{1}{2}(y_1 + y_2)$.

MODEL PROBLEMS

1. Find the coordinates of the midpoint of the line segment which joins the point $R(4, 6)$ and the point $S(8, -2)$.

Solution:

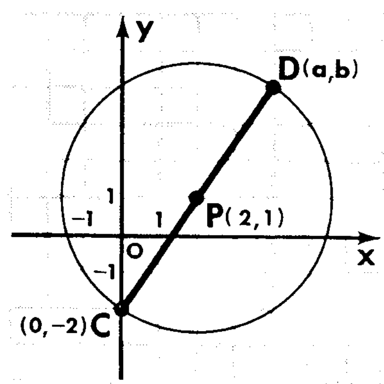


1. Let the point $R(4, 6)$ be (x_1, y_1).
2. Let the point $S(8, -2)$ be (x_2, y_2).
3. Therefore, $x_1 = 4$, $y_1 = 6$, $x_2 = 8$, $y_2 = -2$.
4. Let the midpoint of $\overline{RS}$, point M, be (x_m, y_m).
5. At the midpoint of $\overline{RS}$, $x_m = \frac{1}{2}(x_1 + x_2)$.
6. $x_m = \frac{1}{2}(4 + 8) = \frac{1}{2}(12) = 6$.
7. At the midpoint of $\overline{RS}$, $y_m = \frac{1}{2}(y_1 + y_2)$.
8. $y_m = \frac{1}{2}[6 + (-2)] = \frac{1}{2}[4] = 2$.

Answer: The coordinates of the midpoint of line segment $\overline{RS}$ are $x = 6$, $y = 2$, or $(6, 2)$.

2. $\overline{CD}$ is a diameter of the circle whose center is the point $P(2, 1)$. If the coordinates of C are $(0, -2)$, find the coordinates of D.

Solution:



1. Since P is the center of a circle whose diameter is $\overline{CD}$, P is the midpoint of $\overline{CD}$.
2. Let the point $C(0, -2)$ be (x_1, y_1).
3. Let the point $D(a, b)$ be (x_2, y_2).
4. Therefore, $x_1 = 0$, $y_1 = -2$, $x_2 = a$, $y_2 = b$.
5. Since $P(2, 1)$ is the midpoint of $\overline{CD}$, x_m at the midpoint is 2 and y_m at the midpoint is 1.
6. At the midpoint of $\overline{CD}$, $x_m = \frac{1}{2}(x_1 + x_2)$
7. $2 = \frac{1}{2}(0 + a)$
8. $4 = 0 + a$
9. $4 = a$

10. At the midpoint of $\overline{CD}$, $y_m = \frac{1}{2}(y_1 + y_2)$

11. $1 = \frac{1}{2}(-2 + b)$

12. $2 = -2 + b$

13. $4 = b$

Answer: The coordinates of D are $a = 4$, $b = 4$, or $(4, 4)$.

3. Given the quadrilateral whose vertices are $A(-2, 2)$, $B(1, 4)$, $C(2, 8)$, and $D(-1, 6)$.

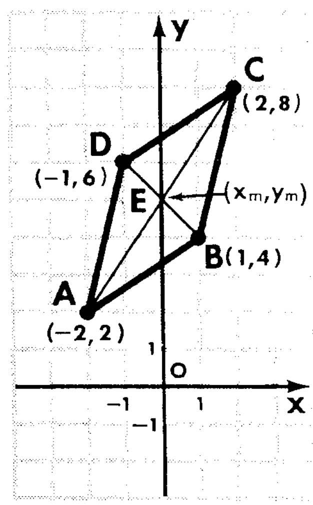
a. Find the coordinates of the midpoint of diagonal $\overline{AC}$.

b. Find the coordinates of the midpoint of diagonal $\overline{BD}$.

c. Show that $ABCD$ is a parallelogram.

Plan: To show that $ABCD$ is a parallelogram, prove that its diagonals bisect each other. They bisect each other when both diagonals have the same midpoint.

Solution:



a. 1. Let $A(-2, 2)$ be (x_1, y_1) and $C(2, 8)$ be (x_2, y_2). Therefore, $x_1 = -2$, $y_1 = 2$, $x_2 = 2$, $y_2 = 8$. Let the midpoint of $\overline{AC}$ be (x_m, y_m).

2. At the midpoint of $\overline{AC}$, $x_m = \frac{1}{2}(x_1 + x_2)$ and $y_m = \frac{1}{2}(y_1 + y_2)$.

3. $x_m = \frac{1}{2}(-2 + 2) = \frac{1}{2}(0) = 0$ and $y_m = \frac{1}{2}(2 + 8) = \frac{1}{2}(10) = 5$.

Answer: The midpoint of diagonal $\overline{AC}$ is the point $(0, 5)$.

b. 1. Let $D(-1, 6)$ be (x_1, y_1) and $B(1, 4)$ be (x_2, y_2). Therefore, $x_1 = -1$, $y_1 = 6$, $x_2 = 1$, $y_2 = 4$. Let the midpoint of $\overline{BD}$ be (x_m, y_m).

2. At the midpoint of $\overline{BD}$, $x_m = \frac{1}{2}(x_1 + x_2)$ and $y_m = \frac{1}{2}(y_1 + y_2)$.

3. $x_m = \frac{1}{2}(-1 + 1) = \frac{1}{2}(0) = 0$ and $y_m = \frac{1}{2}(6 + 4) = \frac{1}{2}(10) = 5$.

Answer: The midpoint of diagonal $\overline{BD}$ is the point $(0, 5)$.

c. 1. Since the point $E(0, 5)$ is the midpoint of each diagonal, the diagonals bisect each other at point E.

2. Since $ABCD$ is a quadrilateral whose diagonals bisect each other, it is a parallelogram.

EXERCISES

1. Find the coordinates of the midpoint of the line segment which joins each of the following pairs of points:
a. (6, 8), (4, 10) *b.* (2, 6), (8, 4) *c.* (2, 11), (7, 6)
d. (5, −6), (−1, 9) *e.* (−4, −8), (−6, −5) *f.* (12, 11), (−1, −2)
g. (2, 0), (5, 9) *h.* (−2, 7), (0, −5) *i.* (0, 0), (8, 10)
j. (2, −7), (−2, 7) *k.* ($5c$, $2c$), (c, $8c$) *l.* (m, 0), (0, n)
2. Find the abscissa of the midpoint of the line segment whose endpoints are:
a. (4, 8), (10, 12) *b.* (8, 7), (6, 3) *c.* (6, −6), (12, −4)
d. (−7, −3), (−5, −7) *e.* (0, 0), (6, 10) *f.* (0, −5), (−7, 0)
g. ($2a$, $3b$), ($4a$, $6b$) *h.* ($-4c$, $2d$), ($8c$, $6d$) *i.* (a, 0), (0, b)
3. Find the ordinate of the midpoint of the line segment whose endpoints are:
a. (6, 10), (8, 6) *b.* (10, 5), (8, 1) *c.* (8, −8), (14, −4)
d. (−5, −9), (−3, −5) *e.* (6, −4), (−9, 1) *f.* (−7, 6), (0, 0)
g. (a, $2b$), ($3a$, $4b$) *h.* ($2c$, $-4d$), ($6c$, $8d$) *i.* (c, 0), (0, d)
4. In a circle whose center is P, A and B are the endpoints of a diameter. Find the coordinates of point P when the coordinates of A and B are respectively:
a. (4, 6), (6, 8) *b.* (5, 7), (7, 8) *c.* (−4, −1), (−8, −4)
d. (−5, −2), (3, 7) *e.* (0, 0), (10, 4) *f.* (8, 0), (−4, 0)
g. ($2a$, b), ($4a$, $3b$) *h.* ($-2c$, $-4d$), ($2c$, $4d$) *i.* (a, 0), (0, b)
5. Find the midpoints of the sides of a triangle whose vertices are:
a. $A(2, 8)$, $B(4, 12)$, $C(6, 4)$ *b.* $A(-5, 2)$, $B(7, 4)$, $C(3, -6)$
c. $D(8, 0)$, $E(0, 5)$, $F(4, 6)$ *d.* $D(-2, 0)$, $E(0, -8)$, $F(0, 0)$
6. Find the midpoints of the sides of a quadrilateral whose vertices are the points:
a. $A(0, 0)$, $B(10, 0)$, $C(7, 5)$, $D(3, 5)$
b. $P(-3, 3)$, $Q(11, 3)$, $R(7, 7)$, $S(1, 7)$
7. M is the midpoint of line segment $\overline{CD}$. The coordinates of point C are (8, 4) and of point $M(8, 10)$. Find the coordinates of point D.
8. In triangle ABC, M is the midpoint of side $\overline{AB}$. The coordinates of A are (6, 10) and of point $M(7, -2)$. Find the coordinates of point B.
9. $\overline{CD}$ is a diameter of a circle whose center is P. If the coordinates of point C are (2, 3) and those of point P are (4, 7), find the coordinates of point D.
10. $\overline{CD}$ is a diameter in a circle whose center is P. If the abscissa of point C is $-8a$ and the abscissa of point P is $2a$, find the abscissa of point D.
11. Given the points $A(-4, 6)$, $B(2, 7)$, and $C(r, s)$. If B is the midpoint of line segment $\overline{AC}$, find the value of r and the value of s.
12. The x-axis is the perpendicular bisector of line segment $\overline{AB}$. Find the

coordinates of point B if the coordinates of point A are:
a. (3, 6) *b.* (5, −3) *c.* (−3, 5) *d.* (−2, −6) *e.* (0, 4) *f.* (0, −2)

13. The y-axis is the perpendicular bisector of line segment $\overline{CD}$. Find the coordinates of point D if the coordinates of point C are:
a. (2, 4) *b.* (3, −2) *c.* (−1, 5) *d.* (−3, −6) *e.* (4, 0) *f.* (−5, 0)

14. The origin is the midpoint of line segment $\overline{EF}$. Find the coordinates of point F if the coordinates of point E are:
a. (4, 3) *b.* (2, −1) *c.* (−5, 2) *d.* (−6, −3) *e.* (6, 0) *f.* (0, −4)

15. In parallelogram $ABCD$ the coordinates of A are (6, 7), and the coordinates of C are (12, 3). What are the coordinates of the point of intersection of the diagonals?

16. The points $A(2, 3)$, $B(7, 5)$, $C(8, 8)$, and $D(3, 6)$ are the vertices of a quadrilateral. (*a*) Show that the diagonals of quadrilateral $ABCD$ bisect each other. (*b*) Show that $ABCD$ is a parallelogram.

17. Show that the points $A(1, -3)$, $B(5, -2)$, $C(8, 2)$, and $D(4, 1)$ are the vertices of a parallelogram.

18. Show that the diagonals of the quadrilateral whose vertices are $A(3, 5)$, $B(6, 4)$, $C(7, 8)$, and $D(4, 12)$ do *not* bisect each other.

19. The points $A(4, 0)$, $B(14, 0)$, and $C(8, 6)$ are the vertices of triangle ABC. Show that the length of the line segment which joins the midpoints of $\overline{CA}$ and $\overline{CB}$ is equal to one-half of AB.

20. The points $A(2, 2)$, $B(6, -6)$, $C(8, 2)$, and $D(4, 4)$ are the vertices of polygon $ABCD$. (*a*) Show that $\overline{LM}$, the line segment which joins the midpoints of $\overline{AD}$ and $\overline{DC}$, is congruent and parallel to $\overline{PN}$, the line segment which joins the midpoints of $\overline{BA}$ and $\overline{BC}$. (*b*) If $\overline{MN}$ and $\overline{LP}$ are also drawn, what type of quadrilateral is $LMNP$?

21. Given quadrilateral $ABCD$ whose vertices are $A(0, 0)$, $B(6, 8)$, $C(16, 8)$, and $D(10, 0)$.
a. Using graph paper, construct quadrilateral $ABCD$.
b. If R is the midpoint of $\overline{AB}$, S the midpoint of $\overline{BC}$, and T the midpoint of $\overline{AD}$:
(1) find the length of $\overline{RS}$.
(2) find the length of $\overline{ST}$.
(3) find the length of $\overline{RT}$.
c. Show that RST is a right triangle.

22. The vertices of quadrilateral $ABCD$ are the points $A(4, 0)$, $B(13, 3)$, $C(12, 6)$, and $D(3, 3)$. (*a*) Find the coordinates of the midpoint of $\overline{AC}$. (*b*) Find the coordinates of the midpoint of $\overline{BD}$. (*c*) Show that $ABCD$ is a parallelogram. (*d*) Find the length of diagonal $\overline{AC}$ and the length of diagonal $\overline{BD}$. (*e*) Show that quadrilateral $ABCD$ is a rectangle.

23. The vertices of a triangle are $R(7, 1)$, $S(2, 1)$, $T(4, 7)$. Find the length of the median from R to $\overline{ST}$.

24. Find the length of each median of a triangle whose vertices are $A(4, 8)$, $B(-2, 6)$, $C(0, -4)$.
25. The vertices of triangle ABC are $A(0, 1)$, $B(6, 1)$, and $C(3, 5)$. (*a*) Show that triangle ABC is isosceles. (*b*) Find the length of the altitude from C to $\overline{AB}$.
26. The vertices of triangle ABC are $A(11, -1)$, $B(13, 10)$, and $C(3, 5)$. (*a*) Show that triangle ABC is isosceles. (*b*) Find the length of the altitude from B to $\overline{AC}$.
27. Given triangle RST with vertices $(-3, -4)$, $(3, 4)$, and $(-5, 0)$ respectively.
    a. Show by means of coordinate geometry that triangle RST is a right triangle.
    b. Show by methods of coordinate geometry that the length of the median to the hypotenuse of triangle RST is equal to one-half the length of the hypotenuse.
28. Point $P(2, 3)$ is the center of a circle.
    a. A and B are the endpoints of a diameter of this circle. If the coordinates of A are $(7, 3)$, find the coordinates of B.
    b. Using coordinate geometry, show that point $C(-1, 7)$ is a point on the circle.
    c. Using coordinate geometry, show that triangle ABC is a right triangle.
29. The vertices of triangle ABC are $A(3, 1)$, $B(9, 9)$, and $C(9, 1)$. (*a*) Show that M, the midpoint of $\overline{AB}$, is equidistant from A, B, and C. (*b*) Show that M is the center of the circle which circumscribes triangle ABC. (*c*) Show that triangle ABC is a right triangle.
30. *a.* Given the vertices $R(9, 10)$, $S(13, 2)$, $T(-3, -6)$, draw triangle RST.
    b. Show that $\triangle RST$ is a right triangle.
    c. Find the coordinates of the center of the circle which circumscribes $\triangle RST$.

5. The Slope of a Line Segment and the Slope of a Line

The Slope of a Line Segment

In Fig. 11–14, we can see that line segment $\overline{DE}$ "rises more steeply" than line segment $\overline{AB}$ because $\overline{DE}$ "rises" 20 ft. vertically over a horizontal distance of 60 ft., whereas $\overline{AB}$ "rises" only 10 ft. vertically over the same horizontal

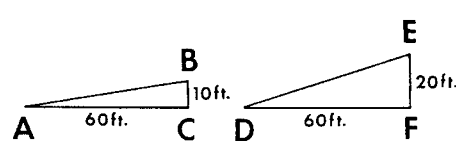


Fig. 11–14

distance of 60 ft. We would say that the *slope* of $\overline{DE}$ is steeper than the slope of $\overline{AB}$. The slope of a line segment involves a comparison of the vertical change with the horizontal change. In mathematics, we define the slope of a line segment in such a way that the general ideas that we have mentioned are expressed more precisely.

Definition. **The *slope, m, of a line segment* whose endpoints are $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, where $x_1 \neq x_2$, is the ratio of the difference of the y-values, $y_2 - y_1$, to the difference of the corresponding x-values, $x_2 - x_1$.**

Hence, in Fig. 11–15, the slope of the line segment whose endpoints are $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad (x_1 \neq x_2)$$

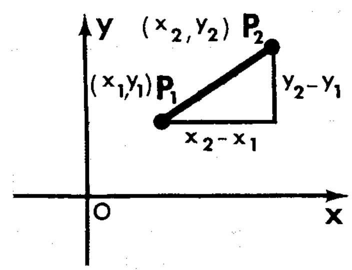


Fig. 11–15

The expression "difference in x-values, $x_2 - x_1$" may be represented by the symbol Δx, read "delta x." Likewise, "the difference in y-values, $y_2 - y_1$" may be represented by Δy, read "delta y." Therefore, we may write the formula for the slope of the line segment whose endpoints are $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ more simply as

$$m = \frac{\Delta y}{\Delta x}$$

Note that in finding the slope of a line segment whose endpoints are two given points, it does not matter which point is represented by (x_1, y_1) and which by (x_2, y_2) since $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$.

We will now see that the slope of a non-vertical line segment may be a positive number or a negative number or zero.

Positive Slope

In Fig. 11–16, reading from left to right, if the segment $\overline{P_1P_2}$ slants upward, then $y_2 - y_1$, or Δy, is a positive number (+3); and $x_2 - x_1$, or Δx, is also a positive number (+2). Since both Δy and Δx are positive numbers, the slope of $\overline{P_1P_2}$, $\frac{\Delta y}{\Delta x}$, must be a positive number.

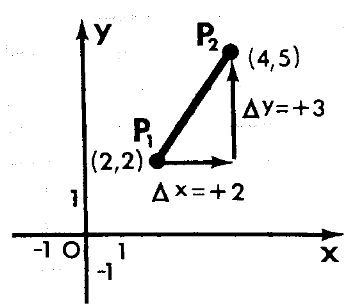


Fig. 11–16

In this case,

$$\text{slope of } \overline{P_1P_2} = m = \frac{\Delta y}{\Delta x} = \frac{+3}{+2} = +\frac{3}{2}$$

Thus, we have illustrated the truth of the following statement:

The slope of a non-vertical line segment that slants upward from left to right is a positive number.

Negative Slope

In Fig. 11–17, reading from left to right, if the segment $\overline{P_1P_2}$ slants downward, then $y_2 - y_1$, or Δy, is a negative number (−3); and $x_2 - x_1$, or Δx, is a positive number (+2). Since Δy is a negative number and Δx is a positive number, the slope of $\overline{P_1P_2}$, $\frac{\Delta y}{\Delta x}$, must be a negative number.

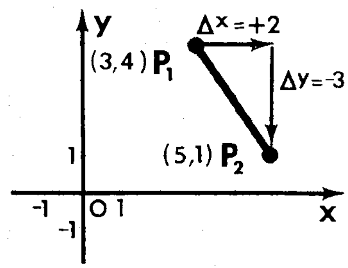


Fig. 11–17

In this case,

$$\text{slope of } \overline{P_1P_2} = m = \frac{\Delta y}{\Delta x} = \frac{-3}{+2} = -\frac{3}{2}$$

Thus, we have illustrated the truth of the following statement:

The slope of a non-vertical line segment that slants downward from left to right is a negative number.

Zero Slope

In Fig. 11–18, $\overline{AB} \parallel x$-axis. Hence, the ordinates of points A and B must be equal. Therefore, we can represent the coordinates of point A by (x_1, y_1) and the coordinates of point B by (x_2, y_1), where $x_1 \neq x_2$.

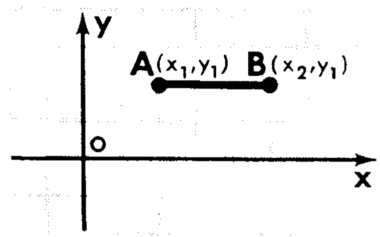


Fig. 11–18

The slope of $\overline{AB} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0.$

Hence, we have proved:

***Theorem 149.* The slope of a line segment parallel to the x-axis (a horizontal line segment) is zero.**

[NOTE. Later, we will see that the slope of the x-axis itself is also zero.]

No Slope

In Fig. 11–19, $\overline{CD} \parallel y$-axis. Hence, the abscissas of points C and D must be equal. Therefore, the coordinates of C may be represented by (x_1, y_1) and the coordinates of D may be represented by (x_1, y_2) where $y_1 \neq y_2$.

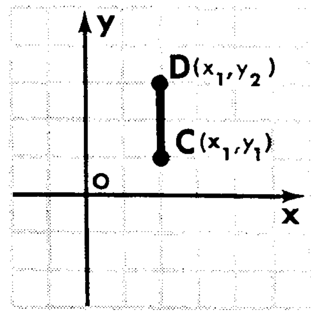


Fig. 11–19

The slope of $\overline{CD} = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_1 - x_1} = \dfrac{y_2 - y_1}{0}$.

The expression $\dfrac{y_2 - y_1}{0}$ is undefined since division by zero is undefined.

Hence, we have proved:

Theorem 150. **A line segment parallel to the *y*-axis (a vertical line segment) has no defined slope.**

[NOTE. Later, we shall see that the y-axis itself has no defined slope.]

The Slope of a Line

Suppose P_1, P_2, and P_3 are points on non-vertical line $\overleftrightarrow{LM}$, as in Fig. 11–20.

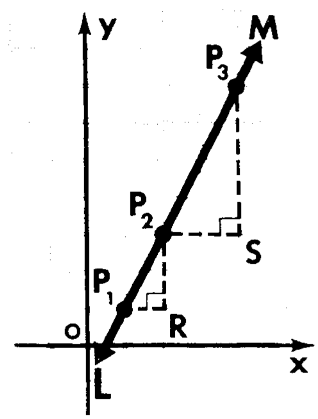


Fig. 11–20

The slope of $\overline{P_1P_2} = \dfrac{P_2R}{P_1R}$.

The slope of $\overline{P_2P_3} = \dfrac{P_3S}{P_2S}$.

Since $\triangle P_1RP_2 \sim \triangle P_2SP_3$, then $\dfrac{P_2R}{P_1R} = \dfrac{P_3S}{P_2S}$.

Therefore, the slope of $\overline{P_1P_2}$ = the slope of $\overline{P_2P_3}$.

Hence, we have outlined the proof of:

Theorem 151. **The slopes of all segments that lie on a non-vertical straight line are equal.**

Definition. **The *slope of a non-vertical line* is equal to the slope of any segment on the line.**

We also say that a vertical line (a line parallel to the y-axis, or the y-axis itself) has no defined slope.

We see, therefore, that although the slope of a non-vertical line segment is a number which is associated with the line segment, it is also the slope of the line which contains the line segment.

It can be shown that two distinct lines that pass through the same point must have unequal slopes.

Recall that collinear points are points that lie on the same straight line. In Fig. 11–21, since points P_1, P_2, and P_3 all lie on $\overleftrightarrow{LM}$, they are collinear points.

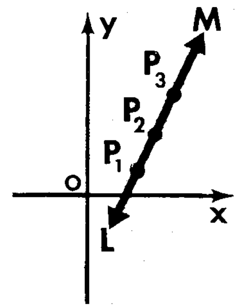


Fig. 11–21

Theorem 152. **Three points are collinear if the slope of the line segment which joins the first point and the second point is equal to**

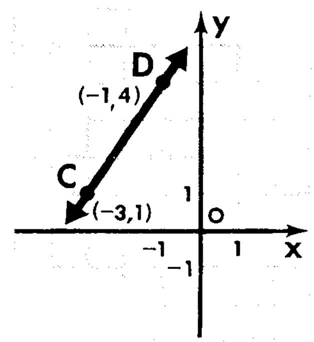
(*a*) the slope of the line segment which joins the first point and the third point, or
(*b*) the slope of the line segment which joins the second point and the third point.

In Fig. 11–21, P_1, P_2, and P_3 are collinear if the slope of $\overline{P_1P_2}$ is equal to the slope of $\overline{P_1P_3}$, or if the slope of $\overline{P_1P_2}$ is equal to the slope of $\overline{P_2P_3}$.

MODEL PROBLEMS

1. Find the slope of the line which passes through the points $C(-3, 1)$ and $D(-1, 4)$.

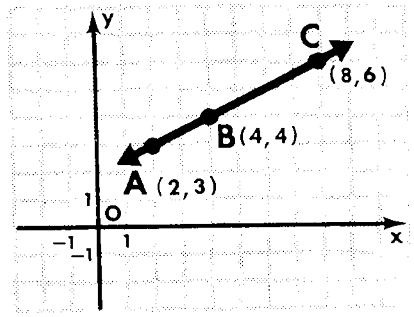
Solution:



1. Let the coordinates of C be (x_1, y_1). Then $x_1 = -3$ and $y_1 = 1$.
2. Let the coordinates of D be (x_2, y_2). Then $x_2 = -1$ and $y_2 = 4$.
3. Slope of $\overleftrightarrow{CD} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4-1}{-1-(-3)}$.
4. Slope of $\overleftrightarrow{CD} = \dfrac{4-1}{-1+3} = \dfrac{3}{2}$.

Answer: Slope of $\overleftrightarrow{CD}$ is $\dfrac{3}{2}$.

2. Show that the points $A(2, 3)$, $B(4, 4)$, and $C(8, 6)$ are collinear.



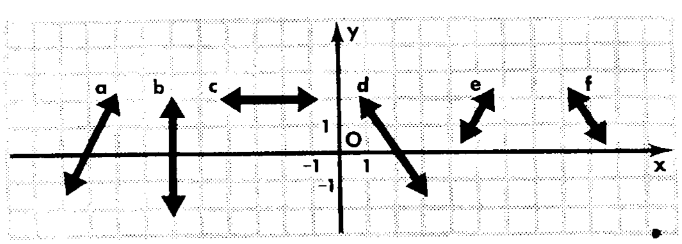
Plan: To show that points A, B, and C are collinear, show that the slope of line segment $\overline{AB}$ is equal to the slope of line segment $\overline{AC}$.

Solution:

1. Let the coordinates of A be (x_1, y_1). Then $x_1 = 2$ and $y_1 = 3$.
2. Let the coordinates of B be (x_2, y_2). Then $x_2 = 4$ and $y_2 = 4$.
3. Slope of line segment $\overline{AB} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4-3}{4-2} = \dfrac{1}{2}$.
4. Let the coordinates of A be (x_1, y_1). Then $x_1 = 2$ and $y_1 = 3$.
5. Let the coordinates of C be (x_2, y_2). Then $x_2 = 8$ and $y_2 = 6$.
6. Slope of line segment $\overline{AC} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{6-3}{8-2} = \dfrac{3}{6} = \dfrac{1}{2}$.
7. Since A is a point on line segment $\overline{AB}$ as well as on line segment $\overline{AC}$, and the slope of $\overline{AB}$ is equal to the slope of $\overline{AC}$, points A, B, and C all lie on the same straight line, $\overleftrightarrow{AC}$, and are therefore collinear.

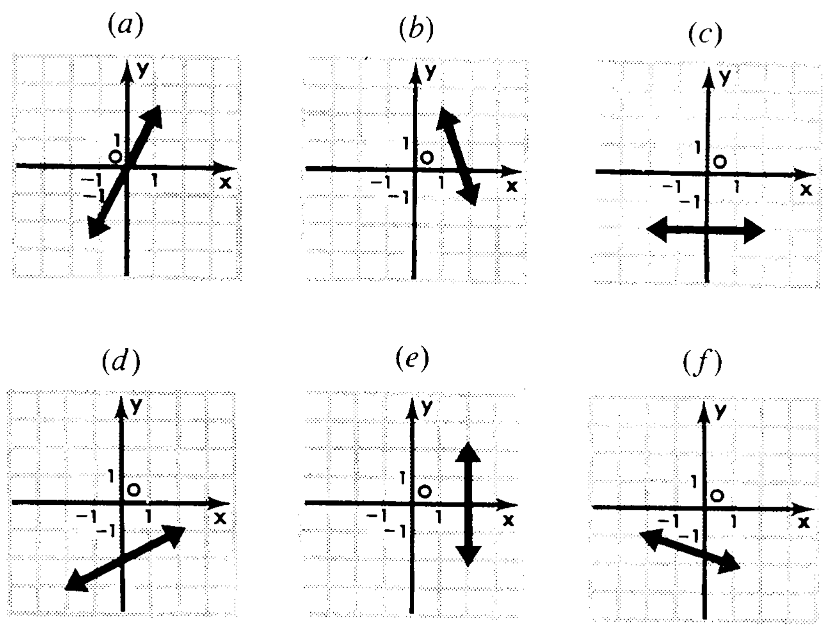
EXERCISES

1. In each part, tell whether the line has a positive slope, a negative slope, a slope of zero, or no slope.



Ex. 1

2. In each part, find the slope of the line; if the line has no slope, indicate that fact.



Ex. 2

3. In each part, plot both points on coordinate graph paper, draw the line segment which joins them, and find the slope of this line segment.
   a. (0, 0) and (9, 3) *b.* (0, 0) and (−2, 4) *c.* (−2, −6) and (0, 0)
   d. (0, 4) and (2, 8) *e.* (−5, 6) and (−1, 0) *f.* (0, −3) and (−6, 0)
   g. (1, 5) and (3, 9) *h.* (2, 3) and (4, 15) *i.* (5, 8) and (4, 3)
   j. (2, 7) and (6, 9) *k.* (4, 6) and (−5, 9) *l.* (5, −2) and (7, −8)
4. Find the slope of the line which passes through the points (3, 5) and (8, 5).
5. Choose the correct answer: The straight line which passes through the points (6, 3) and (2, 3) has (*a*) a slope of 4 (*b*) a slope of zero (*c*) no slope.
6. Find the value of x so that the slope of the line passing through the points (5, 3) and (x, 6) will be 1.
7. Find the value of y so that the slope of the line passing through the points (2, y) and (6, 10) will be $\frac{1}{2}$.
8. Find the value of y so that the slope of the line passing through the points (−8, −2) and (4, y) will be 0.
9. Find the value of x so that the line passing through the points (−4, 8) and (x, −2) will have no slope.
10. In each of the following, determine whether the points are collinear:
   a. (1, 2), (4, 5), (6, 7) *b.* (−1, −4), (2, −2), (8, 2)
   c. (1, 1), (3, 4), (6, 5) *d.* (−2, 6), (0, 2), (1, 0)
   e. (0, 0), (−8, −2), (4, 1) *f.* (−3, 4), (−1, 1), (1, −3)

11. Line $\overleftrightarrow{CD}$ passes through the points $(-4, -2)$ and $(8, 7)$. (*a*) Find the slope of line $\overleftrightarrow{CD}$. (*b*) Tell whether each of the following is *true* or *false:* (1) Line $\overleftrightarrow{CD}$ passes through the point $(0, 1)$. (2) The tangent of the acute angle which line $\overleftrightarrow{CD}$ makes with the x-axis is $\frac{3}{4}$.

6. Parallel Lines and Perpendicular Lines

The number which represents the slope of a line makes it possible for us to state numerical conditions for the parallelism or perpendicularity of two non-vertical lines. We already know that (1) two vertical lines (lines that have no defined slopes) are parallel (2) two horizontal lines (lines that have a slope of 0) are parallel (3) a vertical line (a line with no defined slope) and a horizontal line (a line whose slope is 0) are perpendicular to each other.

Parallel Lines

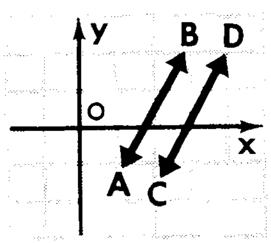


Fig. 11–22

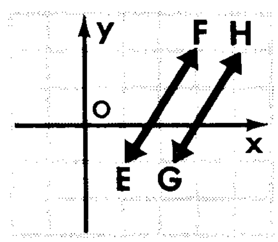


Fig. 11–23

Theorem 153. **If two non-vertical lines are parallel, then their slopes are equal.**

In Fig. 11–22, if $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$, the slope of $\overleftrightarrow{AB} = m_1$, and the slope of $\overleftrightarrow{CD} = m_2$, then $m_1 = m_2$.

Theorem 154. **If two non-vertical lines have equal slopes, then the lines are parallel.**

In Fig. 11–23, if the slope of line $\overleftrightarrow{EF}$ is equal to the slope of line $\overleftrightarrow{GH}$, then $\overleftrightarrow{EF}$ is parallel to $\overleftrightarrow{GH}$.

Perpendicular Lines

Definition. **One number is the *reciprocal* of a second number if the product of the two numbers is 1.**

Thus, $\frac{4}{5}$ is the reciprocal of $\frac{5}{4}$ since $\left(\frac{5}{4}\right)\left(\frac{4}{5}\right) = 1$.

Also, $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$ since $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 1$.

Thus, to find the reciprocal of a fraction, we simply invert the fraction.

***Definition.* One number is the *negative reciprocal* of a second number if the product of the two numbers is −1.**

Thus, $-\frac{1}{3}$ is the negative reciprocal of 3 since $(3)\left(-\frac{1}{3}\right) = -1$.

Also, $-\frac{b}{a}$ is the negative reciprocal of $\frac{a}{b}$ since $\left(\frac{a}{b}\right)\left(-\frac{b}{a}\right) = -1$.

Thus, to find the negative reciprocal of a fraction, we simply invert the fraction and change its sign.

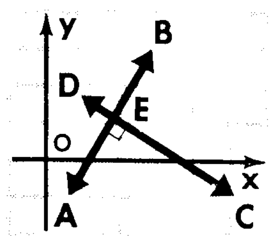


Fig. 11–24

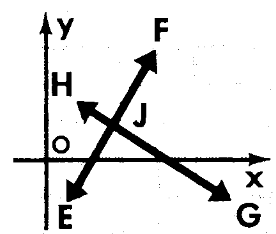


Fig. 11–25

***Theorem 155.* If two non-vertical lines are perpendicular, then the slope of one line is the negative reciprocal of the slope of the other line.**

In Fig. 11–24, if line $\overleftrightarrow{AB}$ whose slope is m_1 is perpendicular to line $\overleftrightarrow{CD}$ whose slope is m_2, then $m_1m_2 = -1$, or $m_1 = -\frac{1}{m_2}$.

***Theorem 156.* If the slope of one line is the negative reciprocal of the slope of a second line, then the lines are perpendicular.**

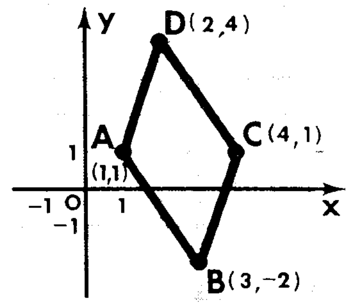
In Fig. 11–25, if the slope of line $\overleftrightarrow{EF}$ is m_1, the slope of line $\overleftrightarrow{GH}$ is m_2, and $m_1m_2 = -1$, then $\overleftrightarrow{EF}$ is perpendicular to $\overleftrightarrow{GH}$.

KEEP IN MIND

1. To show that two non-vertical lines are parallel, show that the slopes of the lines are equal.
2. To show that two non-vertical lines are perpendicular, show that the slope of one line is the negative reciprocal of the slope of the other line.

MODEL PROBLEMS

1. Show, by means of slopes, that the quadrilateral whose vertices are the points $A(1, 1)$, $B(3, -2)$, $C(4, 1)$, and $D(2, 4)$ is a parallelogram.

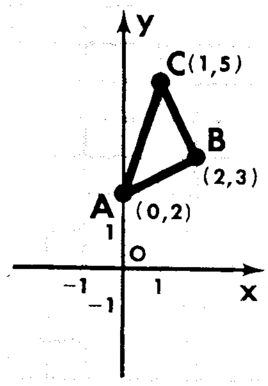


Plan: To show that $ABCD$ is a parallelogram, show that its opposite sides are parallel. To show that two sides are parallel, show that their slopes are equal.

Solution:

1. Let the coordinates of A be (x_1, y_1). Then $x_1 = 1$ and $y_1 = 1$.
2. Let the coordinates of B be (x_2, y_2). Then $x_2 = 3$ and $y_2 = -2$.
3. Slope of $\overline{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 1}{3 - 1} = \frac{-3}{2} = -\frac{3}{2}$.
4. Let the coordinates of D be (x_1, y_1). Then $x_1 = 2$ and $y_1 = 4$.
5. Let the coordinates of C be (x_2, y_2). Then $x_2 = 4$ and $y_2 = 1$.
6. Slope of $\overline{DC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 4}{4 - 2} = \frac{-3}{2} = -\frac{3}{2}$.
7. Since slope of $\overline{AB}$ = slope of $\overline{DC}$, $\overline{AB}$ is parallel to $\overline{DC}$.
8. Let the coordinates of B be (x_1, y_1). Then $x_1 = 3$ and $y_1 = -2$.
9. Let the coordinates of C be (x_2, y_2). Then $x_2 = 4$ and $y_2 = 1$.
10. Slope of $\overline{BC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-2)}{4 - 3} = \frac{1 + 2}{4 - 3} = \frac{3}{1} = 3$.
11. Let the coordinates of A be (x_1, y_1). Then $x_1 = 1$ and $y_1 = 1$.
12. Let the coordinates of D be (x_2, y_2). Then $x_2 = 2$ and $y_2 = 4$.
13. Slope of $\overline{AD} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{2 - 1} = \frac{3}{1} = 3$.
14. Since slope of $\overline{BC}$ = slope of $\overline{AD}$, $\overline{BC}$ is parallel to $\overline{AD}$.
15. Since the opposite sides of quadrilateral $ABCD$ are parallel, quadrilateral $ABCD$ is a parallelogram.

2. Show, by means of slopes, that the triangle whose vertices are $A(0, 2)$, $B(2, 3)$, and $C(1, 5)$ is a right triangle.



Plan: To show that triangle ABC is a right triangle, show that two of its sides are perpendicular by showing that the slope of one side is the negative reciprocal of the slope of the other side.

Solution:

1. Let the coordinates of A be (x_1, y_1). Then $x_1 = 0$ and $y_1 = 2$.
2. Let the coordinates of B be (x_2, y_2). Then $x_2 = 2$ and $y_2 = 3$.
3. Slope of $\overline{AB} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-2}{2-0} = \dfrac{1}{2}$.
4. Let the coordinates of B be (x_1, y_1). Then $x_1 = 2$ and $y_1 = 3$.
5. Let the coordinates of C be (x_2, y_2). Then $x_2 = 1$ and $y_2 = 5$.
6. Slope of $\overline{BC} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{5-3}{1-2} = \dfrac{2}{-1} = -2$.
7. The slope of $\overline{AB}$ is the negative reciprocal of the slope of $\overline{BC}$ because $(\frac{1}{2})(-2) = -1$.
8. Therefore, $\overline{AB}$ is perpendicular to $\overline{BC}$.
9. Triangle ABC is a right triangle because it contains right angle ABC.

EXERCISES

1. Show that the line which joins the points (1, 3) and (5, 6) is parallel to the line which joins the points (5, 1) and (9, 4).

2. The vertices of quadrilateral $ABCD$ are $A(2, 3)$, $B(8, 5)$, $C(9, 9)$, and $D(3, 7)$. (*a*) Using graph paper, plot these vertices and draw the quadrilateral. (*b*) Find the slope of each side of the quadrilateral. (*c*) Show that $ABCD$ is a parallelogram.

3. Using the formula for the slope of a line, show that the points (−2, 3), (2, 7), (8, 5), and (4, 1) are the vertices of a parallelogram.

4. (*a*) Show by using the formula for the slope of a line that the points

(3, 1), (6, 3), (10, 0), and (7, −2) are the vertices of a parallelogram. (*b*) Show the same conclusion by showing that the diagonals of the quadrilateral bisect each other.

5. (*a*) Show by using the formula for the slope of a line that the points (−4, 0), (−1, 3), (3, 1), and (0, −2) are the vertices of a parallelogram. (*b*) Show the same conclusion by showing that both pairs of opposite sides of the quadrilateral are equal in length.
6. $A(1, 3)$, $B(7, 5)$, $C(9, -3)$ are the vertices of triangle ABC. E is the midpoint of $\overline{AB}$ and F is the midpoint of $\overline{BC}$. Show that $\overline{EF}$ is parallel to $\overline{AC}$ and $EF = \frac{1}{2}AC$.
7. The vertices of quadrilateral $ABCD$ are the points $A(-2, -2)$, $B(4, 0)$, $C(2, 4)$, and $D(-6, 6)$. E is the midpoint of $\overline{AB}$, F is the midpoint of $\overline{BC}$, G is the midpoint of $\overline{CD}$, and H is the midpoint of $\overline{AD}$. (*a*) Find the coordinates of E, F, G, and H. (*b*) Show that $\overline{EF}$ is parallel to $\overline{GH}$. (*c*) Show that $EF = GH$. (*d*) Show that $EFGH$ is a parallelogram.
8. The coordinates of the vertices of quadrilateral $ABCD$ are $A(0, 5)$, $B(3, 4)$, $C(0, -5)$, and $D(-3, -4)$.
   a. Using graph paper, draw quadrilateral $ABCD$.
   b. Show that $ABCD$ is a parallelogram.
   c. Show that $ABCD$ is a rectangle.
9. Find the slope of a line which is perpendicular to a line whose slope is:
   a. $\frac{3}{4}$ *b.* $\frac{5}{9}$ *c.* $-\frac{2}{3}$ *d.* 4 *e.* −3 *f.* $1\frac{1}{4}$ *g.* .1
10. Find the slope of a line which is perpendicular to the line which passes through the points:
    a. (5, 6) and (8, 11) *b.* (1, 4) and (3, −7) *c.* (−2, −3) and (0, 3)
11. Show that the line which passes through the points (2, 3) and (5, 1) is perpendicular to the line which passes through the points (5, 4) and (1, −2).

In 12 and 13, the given points A, B, and C are the vertices of a triangle. (*a*) Find the slope of each side of the triangle. (*b*) Find the slope of each altitude of the triangle.

12. $A(1, 1)$, $B(5, 2)$, $C(3, 4)$
13. $A(-3, -2)$, $B(3, -1)$, $C(5, 4)$

14. Determine by means of slopes which of the following groups of points are the vertices of a right triangle. Check your answer by using the distance formula.
    a. (2, 2), (4, 1), (4, 6) *b.* (2, 5), (−4, 3), (−3, 0)
    c. (1, 1), (4, 4), (7, 2) *d.* (1, 1), (3, 0), (0, −4)
15. The vertices of a quadrilateral are (3, 1), (5, 6), (7, 6), and (10, 2). Show that the diagonals of the quadrilateral are perpendicular to each other.

16. *Given:* $A(0, 4)$, $B(-5, 0)$, and $C(3, 0)$

a. Find the length of $\overline{AC}$.

b. Find the coordinates of point D so that $ABCD$ is a parallelogram.

c. Find the coordinates of point E so that $ABEC$ is a parallelogram.

d. What is the greatest number of parallelograms possible with the points A, B, and C as three of the vertices?

17. The points $A(2, 1)$, $B(9, 4)$, $C(5, 8)$ are the vertices of triangle ABC. Show that the median from A is perpendicular to $\overline{BC}$.

18. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. The slope of $\overleftrightarrow{AB}$ is $\frac{3}{4}$. The slope of $\overleftrightarrow{CD}$ is $\frac{9}{x}$. Find x.

19. $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$. The slope of $\overleftrightarrow{EF}$ is $-\frac{2}{3}$. The slope of $\overleftrightarrow{GH}$ is $\frac{8}{x-6}$. Find x.

20. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. The slope of $\overleftrightarrow{AB}$ is $\frac{3}{5}$. The slope of $\overleftrightarrow{CD}$ is $\frac{10}{x}$. Find x.

21. $\overleftrightarrow{PQ} \perp \overleftrightarrow{RS}$. The slope of $\overleftrightarrow{PQ}$ is $\frac{x-1}{4}$. The slope of $\overleftrightarrow{RS}$ is $\frac{8}{3}$. Find x.

22. The vertices of parallelogram $ABCD$ have the following coordinates: $A(-2, 4)$, $B(2, 6)$, $C(7, 2)$, $D(x, 0)$. (*a*) Find the slope of $\overline{AB}$. (*b*) Express the slope of $\overline{DC}$ in terms of x. (*c*) Using the results found in answer to *a* and *b*, find the value of x.

23. (*a*) Find the slope of line $\overleftrightarrow{CD}$ which passes through the points $(2, 3)$ and $(10, 9)$. (*b*) Without using graph paper, show that the point $(-2, 0)$ lies on line $\overleftrightarrow{CD}$. (*c*) If the point $P(14, y)$ lies on $\overleftrightarrow{CD}$, find the value of y.

24. The points $A(-1, 4)$, $B(-2, 1)$, $C(4, 3)$, and $D(t, 5)$ are the vertices of trapezoid $ABCD$ whose bases are $\overline{BC}$ and $\overline{AD}$. (*a*) Find the slope of $\overline{BC}$. (*b*) Express, in terms of t, the slope of $\overline{AD}$. (*c*) Using the results found in answer to *a* and *b*, find the value of t. (*d*) Show that $ABCD$ is not an isosceles trapezoid.

7. Proving Theorems by Using Coordinate Geometry

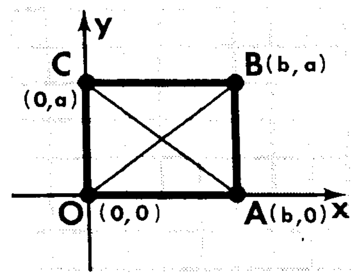
We can use coordinate geometry to prove some of the theorems that we have already proved in our study of plane geometry. When we prepare the figure for the discussion of the proof of a theorem, it is best to place it in a position which will simplify the details of the proof. In the case of a polygon, it is usually helpful to place one vertex at the origin and one side along the positive half-line of the x-axis. In the case of a circle, the center is usually placed at the origin.

KEEP IN MIND

1. To prove that line segments are congruent or equal in length, show that the length of each segment is represented by the same number symbol.
2. To prove that lines are parallel, show that the slopes of the lines are represented by the same number symbol.
3. To prove that lines are perpendicular, show that the product of the number symbols which represent their slopes is -1.
4. To prove that line segments bisect each other show that the same ordered pair of number symbols represents the midpoint of each line segment.

MODEL PROBLEMS

1. Prove that the diagonals of a rectangle are congruent.



Place the rectangle so that one vertex, O, is at the origin, side $\overline{OA}$ is on the x-axis, and vertex B is in quadrant I. The coordinates of the vertices can be represented by $O(0, 0)$, $A(b, 0)$, $B(b, a)$, $C(0, a)$.

Given: Rectangle $OABC$.

To prove: $\overline{OB} \cong \overline{AC}$.

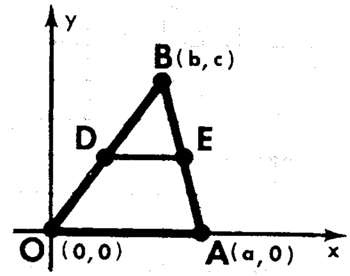
Plan: Use the distance formula to find the lengths of $\overline{OB}$ and $\overline{AC}$. Show that these lengths are equal.

Proof:

1. Let the coordinates of O be (x_1, y_1). Then $x_1 = 0$ and $y_1 = 0$.
2. Let the coordinates of B be (x_2, y_2). Then $x_2 = b$ and $y_2 = a$.
3. $OB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(b - 0)^2 + (a - 0)^2} = \sqrt{b^2 + a^2}$.
4. Let the coordinates of C be (x_1, y_1). Then $x_1 = 0$ and $y_1 = a$.
5. Let the coordinates of A be (x_2, y_2). Then $x_2 = b$ and $y_2 = 0$.
6. $AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(b - 0)^2 + (0 - a)^2} = \sqrt{b^2 + a^2}$.
7. Since $OB = \sqrt{b^2 + a^2}$ and $AC = \sqrt{b^2 + a^2}$, $OB = AC$, and $\overline{OB} \cong \overline{AC}$.

2. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side.

Place the triangle so that one vertex, O, is at the origin, side $\overline{OA}$ is on the x-axis, and vertex B is in quadrant I. The coordinates of the vertices can be represented by $O(0, 0)$, $A(a, 0)$, $B(b, c)$.



Given: In triangle ABO, D is the midpoint of $\overline{OB}$ and E is the midpoint of $\overline{AB}$.

To prove: $\overline{DE} \parallel \overline{OA}$.

Plan: To prove that $\overline{DE}$ and $\overline{OA}$ are parallel, show that the slopes of these line segments are equal.

1. Since D is the midpoint of $\overline{OB}$, x at $D = \frac{1}{2}(0 + b) = \frac{1}{2}b$, and y at $D = \frac{1}{2}(0 + c) = \frac{1}{2}c$. Therefore, the coordinates of D are $(\frac{1}{2}b, \frac{1}{2}c)$.
2. Since E is the midpoint of $\overline{AB}$, x at $E = \frac{1}{2}(b + a)$, and y at $E = \frac{1}{2}(c + 0) = \frac{1}{2}c$. Therefore, the coordinates of E are $[\frac{1}{2}(b + a), \frac{1}{2}c]$.
3. Slope of $DE = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\frac{1}{2}c - \frac{1}{2}c}{\frac{1}{2}(b + a) - \frac{1}{2}b} = \dfrac{0}{\frac{1}{2}a} = 0. \qquad (a \neq 0)$
4. Slope of $\overline{OA} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0 - 0}{a - 0} = \dfrac{0}{a} = 0. \qquad (a \neq 0)$
5. Since the slope of $\overline{DE}$ is equal to the slope of $\overline{OA}$, $\overline{DE}$ is parallel to $\overline{OA}$.

EXERCISES

In 1–6, prove the theorems using coordinate geometry.

1. The diagonals of a square are congruent.

2. The length of the line segment that joins the midpoints of two sides of a triangle is equal to one-half the length of the third side.

3. The midpoint of the hypotenuse of a right triangle is equidistant from the vertices of the triangle. [*Hint:* Represent the vertices of the triangle by (0, 0), (0, 2a), and (2b, 0).]

4. The line segments joining the midpoints of the sides of a square, taken in order, form an equilateral quadrilateral.

5. The median to the base of an isosceles triangle is perpendicular to the base of the triangle.

6. The length of the median of a trapezoid is equal to one-half the sum of the lengths of its bases.

7. The vertices of quadrilateral $RSTV$ are $R(0, 0)$, $S(a, 0)$, $T(a+b, c)$, and $V(b, c)$. (*a*) Find the slope of $\overline{RV}$ and the slope of $\overline{ST}$. (*b*) Show that $RSTV$ is a parallelogram.
8. The vertices of quadrilateral $ABCD$ are $A(0, 0)$, $B(r, s)$, $C(r, s+t)$, and $D(0, t)$. (*a*) Represent the slope of $\overline{AB}$ and the slope of $\overline{CD}$. (*b*) Represent the length of $\overline{AB}$ and the length of $\overline{CD}$. (*c*) Show that $ABCD$ is a parallelogram.
9. The vertices of triangle RST are $R(0, 0)$, $S(a, b)$, and $T(c, d)$. (*a*) Find the coordinates of E, the midpoint of $\overline{TR}$. (*b*) Find the coordinates of F, the midpoint of $\overline{TS}$. (*c*) Find the slope of $\overline{EF}$ and the slope of $\overline{RS}$. (*d*) Show that $\overline{EF}$ is parallel to $\overline{RS}$.
10. The vertices of triangle ABC are $A(0, 0)$, $B(4a, 0)$, and $C(2a, 2b)$. (*a*) Find the coordinates of D, the midpoint of $\overline{AC}$. (*b*) Find the coordinates of E, the midpoint of $\overline{BC}$. (*c*) Show that $AB = 2DE$.
11. The vertices of quadrilateral $ABCD$ are $A(0, 0)$, $B(a, 0)$, $C(a, b)$, and $D(0, b)$. (*a*) Show that $ABCD$ is a parallelogram. (*b*) Show that diagonal $\overline{AC}$ is congruent to diagonal $\overline{BD}$. (*c*) Show that quadrilateral $ABCD$ is a rectangle.
12. The vertices of $\triangle RST$ are $R(0, 0)$, $S(2a, 2b)$, and $T(4a, 0)$. The midpoints of $\overline{RS}$, $\overline{ST}$, and $\overline{TR}$ are L, M, and N respectively.
    a. Express the coordinates of L, M, and N in terms of a and b.
    b. Express the lengths of the medians from R, S, and T in terms of a and b.
    c. $\triangle RST$ must be (1) equilateral (2) right (3) isosceles (4) scalene.

8. Graphing a Linear Equation

In your algebra course, you learned that an ordered pair of numbers which satisfies an equation is a *solution* of the equation. For example, $(3, 1)$ is a solution of the equation $x + 3y = 6$ because when x is replaced by 3 and y is replaced by 1 we obtain $3 + 3(1) = 6$, which is a true statement. The set of all ordered pairs of numbers which are the solutions of the equation $x + 3y = 6$ is called the *solution set* of $x + 3y = 6$. The equation $x + 3y = 6$ is an example of a first-degree equation in two variables. The general form of such an equation is $ax + by = c$, where a and b are not both 0.

You have also learned that:

***Definition*. The *graph of an equation* is the graph of the solution set of that equation.**

From this definition, we can state that:

1. Any ordered pair of numbers that is a member of the solution set of an equation (that is, the ordered pair satisfies the equation) represents the coordinates of a point of the graph of that equation.
2. Any point on the graph of an equation has as its coordinates an ordered pair of numbers which is a member of the solution set of that equation (that is, the ordered pair satisfies that equation).

In this book, we will assume the truth of the following two statements:

Postulate 60. **The graph of every first-degree equation in two variables of the form $ax + by = c$, where a and b are not both 0, is a straight line.**

We will soon see that the graph of $x + 3y = 6$ is a straight line.

Postulate 61. **Every graph which is a straight line is the graph of a first-degree equation of the form $ax + by = c$, where a and b are not both 0.**

Therefore, a first-degree equation in two variables is called a *linear equation*. For example, $x + y = 8$ and $y = 2x - 1$ are linear equations.

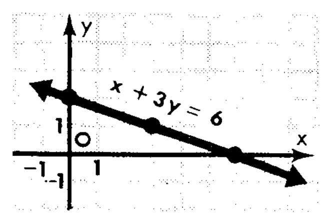
The following model problems will help you recall some methods that are used to graph linear equations.

MODEL PROBLEMS

1. Draw the graph of $x + 3y = 6$.

Solution:

1. Make a table of values by assuming three convenient values for x, such as 0, 3, and 6. Substitute these values in the equation $x + 3y = 6$ to find the corresponding y-values.



If $x = 0$, then $0 + 3y = 6$; $y = 2$
If $x = 3$, then $3 + 3y = 6$; $y = 1$
If $x = 6$, then $6 + 3y = 6$; $y = 0$

x	y
0	2
3	1
6	0

2. Plot the points represented by the number pairs (0, 2), (3, 1), and (6, 0).
3. Draw a straight line through the three points.

In the previous graph of the equation $x + 3y = 6$, the value of y at the point where the graph intersects the *y-axis* is 2. We therefore say that the *y-intercept* is 2. Notice that the value of x at this point is 0. Also, the value of x at the point where the graph intersects the *x-axis* is 6. We therefore say that the *x-intercept* is 6. Notice that the value of y at this point is 0.

Procedure. To find the x-intercept and y-intercept of the graph of an equation:

1. Substitute 0 for y in the given equation to find the x-intercept.
2. Substitute 0 for x in the given equation to find the y-intercept.

2. Find (a) the x-intercept and (b) the y-intercept of the graph of the equation $3x - 2y = 12$.

Solution: (a) To find the x-intercept, let $y = 0$ in the equation $3x - 2y = 12$

$$3x - 2(0) = 12,\ 3x - 0 = 12,\ x = 4$$

(b) To find the y-intercept, let $x = 0$ in the equation $3x - 2y = 12$

$$3(0) - 2y = 12,\ 0 - 2y = 12,\ y = -6$$

Answer: x-intercept = 4, y-intercept = -6.

EXERCISES

1. Draw the graphs of the following equations:

a. $y = 2x$	*b.* $y = 5x$	*c.* $y = -3x$	*d.* $y = -x$
e. $x = 2y$	*f.* $x = 3y$	*g.* $x = -y$	*h.* $x = \frac{1}{2}y$
i. $y = x + 3$	*j.* $y = 2x - 1$	*k.* $y = 3x + 1$	*l.* $y = -2x + 4$
m. $x + y = 8$	*n.* $y + x = 4$	*o.* $x - y = 5$	*p.* $y - x = 6$
q. $2x + y = 10$	*r.* $x + 3y = 12$	*s.* $x - 2y = 6$	*t.* $y - 3x = -5$
u. $2x + 3y = 6$	*v.* $3x + 4y = 12$	*w.* $3x - 2y = -6$	*x.* $4x - 3y = -12$

2. Which of the following ordered pairs of numbers are members of the solution set of the equation (satisfy) $2x - y = 6$? (a) $(4, -2)$ (b) $(2, 2)$ (c) $(4, 2)$ (d) $(0, 6)$ (e) $(3, 0)$

3. What are the coordinates of the point at which the graph of $2x - 3y = 8$ intersects the x-axis?

4. Find the x-intercept and y-intercept of each equation in exercise 1.

In 5–10, state whether the given line passes through the given point.

5. $x + y = 7$, $(4, 3)$

6. $x - y = 5$, $(9, 4)$

7. $2y + x = 7$, $(1, 3)$

8. $3x - 2y = 8$, $(2, -1)$

9. $4x + y = 10$, $(2, -2)$

10. $2y = 3x - 5$, $(-1, -4)$

In 11–13, a point is to lie on the given line. Find its abscissa if its ordinate is the number indicated.

11. $x + y = 12$, (5) **12.** $2x - y = 8$, (−2) **13.** $3x + 2y = 24$, (3)

In 14–16, a point is to lie on the given line. Find its ordinate if its abscissa is the number indicated.

14. $x + 2y = 9$, (3) **15.** $4x - y = 7$, (−1) **16.** $2x + 3y = 5$, (−2)

In 17–20, find the value of k so that the given line will pass through the given point.

17. $x + y = k$, (2, 5) **18.** $x - y = k$, (5, −3)

19. $4x + y = k$, (−1, −3) **20.** $5y - 2x = k$, (−2, 1)

9. Writing an Equation of a Straight Line

The Point–Slope Form

Now we will learn how to write an equation of the line which passes through a given fixed point $P_1(x_1, y_1)$ and which has a given slope, m (Fig. 11–26).

Let us represent any other point on the line as $P(x, y)$. $(x \neq x_1)$

The slope of the line, $m = \dfrac{y - y_1}{x - x_1}$, or

$$y - y_1 = m(x - x_1)$$

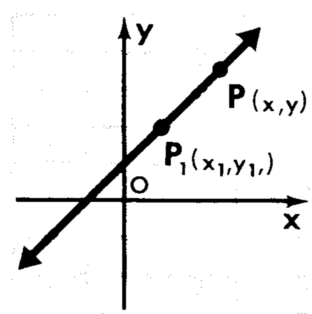


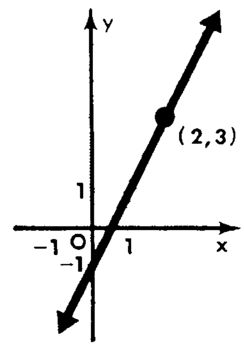
Fig. 11–26

Theorem 157. **An equation of the line passing through the point (x_1, y_1) and having the slope m is $y - y_1 = m(x - x_1)$.**

The equation $y - y_1 = m(x - x_1)$ is called the ***point-slope form*** of an equation of a line.

MODEL PROBLEMS

1. Write an equation of the straight line whose slope is 2 and which passes through the point (2, 3).



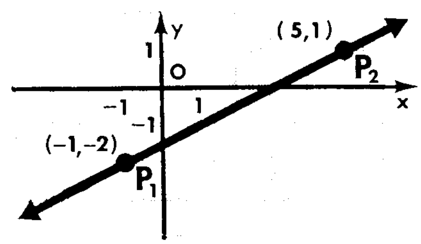
Solution:

1. Since the coordinates of the given point are (2, 3), $x_1 = 2$ and $y_1 = 3$.
2. Since the given slope is 2, $m = 2$.
3. The point-slope form of an equation of the line is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= 2(x - 2) \\ y - 3 &= 2x - 4 \\ y &= 2x - 1 \end{aligned}$$

Answer: $y = 2x - 1$.

2. Write an equation of the line which passes through the points $P_1(-1, -2)$ and $P_2(5, 1)$.



Solution:

1. Since we know two points on the line, we can find the slope of a segment of the line which is the same as the slope of the line. Then we can use the point-slope form of an equation of a line.

[The solution continues on the next page.]

2. Slope of the line. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ $\quad x_1 = -1,\ y_1 = -2.$
$x_2 = 5,\ y_2 = 1.$

$$m = \frac{1 - (-2)}{5 - (-1)} = \frac{1 + 2}{5 + 1} = \frac{3}{6} = \frac{1}{2}$$

3. Now we will write an equation of the line which passes through the point (5, 1) and has a slope of $\frac{1}{2}$.

4. $y - y_1 = m(x - x_1) \qquad m = \frac{1}{2},\ x_1 = 5,\ y_1 = 1.$
$y - 1 = \frac{1}{2}(x - 5)$
$y - 1 = \frac{1}{2}x - \frac{5}{2}$
$y = \frac{1}{2}x - \frac{3}{2}$

Answer: $y = \frac{1}{2}x - \frac{3}{2}$, or $2y = x - 3$.

The Slope–Intercept Form

Let us write an equation of a line whose y-intercept is b and whose slope is m (Fig. 11–27).

If the y-intercept is b, the line passes through the point $(0, b)$.

If we use the point-slope form of an equation of a straight line, we have:

$y - y_1 = m(x - x_1) \qquad \text{slope} = m,\ x_1 = 0,\ y_1 = b.$
$y - b = m(x - 0)$
$y - b = mx$
$y = mx + b$

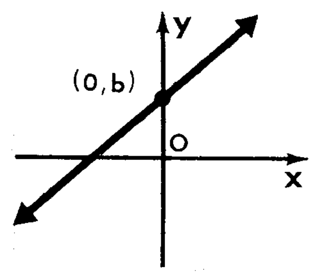


Fig. 11–27

Theorem 158. **An equation of a line whose slope is m and whose y-intercept is b is $y = mx + b$.**

The equation $y = mx + b$ is called the ***slope-intercept form*** of an equation of a line.

Hence, when an equation of a line is expressed in the form $y = mx + b$, then m, the coefficient of x, represents the slope of the line and b represents the y-intercept of the line.

For example, in the equation $y = 3x + 4$, the slope of the line is 3 and the y-intercept is 4.

MODEL PROBLEMS

1. Write an equation of the line whose slope is $\frac{1}{3}$ and y-intercept is -2.

Solution:

1. Use the slope-intercept form of an equation of a line.
2. $y = mx + b$ $\qquad m = \frac{1}{3}$, $b = -2$.
3. $y = \frac{1}{3}x - 2$, or $3y = x - 6$

Answer: $3y = x - 6$.

2. Find the slope and y-intercept of the line whose equation is $4x + 2y = 5$.

Solution:

1. Transform the equation $4x + 2y = 5$ to the form $y = mx + b$.
2. $4x + 2y = 5$
3. $2y = -4x + 5$
4. $y = -2x + \frac{5}{2}$ $\qquad$ Hence, the slope $= -2$, the y-intercept $= \frac{5}{2}$.

Answer: Slope $= -2$; y-intercept $= \frac{5}{2}$.

3. Write an equation of the line which is parallel to the line $6x + 3y = 4$ and whose y-intercept is -6.

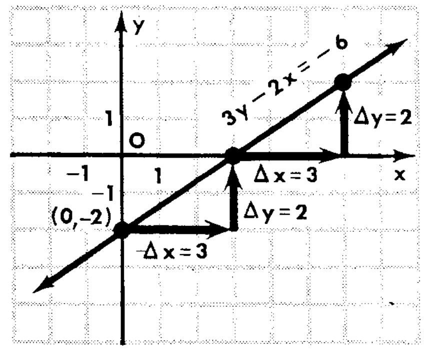
Solution:

1. Transform the equation $6x + 3y = 4$ to the slope-intercept form in order to find its slope.
2. $6x + 3y = 4$
3. $3y = -6x + 4$
4. $y = -2x + \frac{4}{3}$ $\qquad$ [The slope of this line is -2.]
5. The slope of the line $6x + 3y = 4$ is -2.
6. Hence, the slope of a line parallel to the line $6x + 3y = 4$ is also -2.
7. The y-intercept of the required line is -6.
8. Use the slope-intercept form of an equation of a line.
9. $y = mx + b$ $\qquad m = -2$, $b = -6$.
10. $y = -2x - 6$

Answer: $y = -2x - 6$.

4. Draw the graph of $3y - 2x = -6$ using its slope and y-intercept.

Solution:



1. Transform the equation to the slope-intercept form.
2. $3y - 2x = -6$
3. $3y = 2x - 6$
4. $y = \frac{2}{3}x - 2$
5. Slope $= \frac{2}{3}$; y-intercept $= -2$.
6. Since the y-intercept of the line is -2, the point $(0, -2)$ is on its graph.
7. Plot the point $(0, -2)$ on the graph.
8. Since slope $= \dfrac{\Delta y \longrightarrow 2}{\Delta x \longrightarrow 3}$, when x increases 3 units, y increases 2 units. Hence, start at the point $(0, -2)$ and move 3 units to the right and 2 units up to plot a second point. Repeat these movements to plot a third point.
9. Draw the straight line which passes through the three points plotted.

EXERCISES

1. Write an equation of the straight line which has the given slope, m, and which passes through the given point.

a. $m = 3$, $(1, 5)$ *b.* $m = 2$, $(-3, 5)$ *c.* $m = \frac{1}{2}$, $(-2, -3)$
d. $m = \frac{2}{3}$, $(-1, 4)$ *e.* $m = -\frac{2}{5}$, $(0, 0)$ *f.* $m = -\frac{4}{3}$, $(-2, 0)$

2. Through the given point, draw the graph of a line with the given slope m.

a. $(0, 0)$, $m = 2$ *b.* $(1, 3)$, $m = 3$ *c.* $(2, -5)$, $m = 4$
d. $(4, 6)$, $m = \frac{2}{3}$ *e.* $(-4, 5)$, $m = \frac{1}{2}$ *f.* $(-3, -4)$, $m = -2$
g. $(1, -5)$, $m = -1$ *h.* $(2, 4)$, $m = -\frac{3}{2}$ *i.* $(-2, 3)$, $m = -\frac{1}{2}$

3. Write an equation of the line which passes through the given points.

a. $(1, 5)$ and $(5, 13)$ *b.* $(0, 3)$ and $(2, 9)$
c. $(-1, 3)$ and $(1, -1)$ *d.* $(0, 0)$ and $(-2, -4)$
e. $(-4, -1)$ and $(-1, 11)$ *f.* $(12, -5)$ and $(-4, -1)$

4. Write an equation of the straight line whose slope and y-intercept are respectively:

a. 4 and 5 *b.* 2 and -5 *c.* -3 and 2
d. $\frac{3}{4}$ and 4 *e.* $-\frac{2}{3}$ and -1 *f.* $-\frac{5}{3}$ and 4

5. For each of the following lines, find the slope and the y-intercept:
a. $y = 3x + 1$ *b.* $y = x - 4$ *c.* $y = -x$
d. $y = \frac{1}{4}x + 5$ *e.* $2x + y = 9$ *f.* $4y - 2x = 16$

6. Determine the coordinates of the point at which each of the following lines intersects the y-axis:
a. $y = x + 1$ *b.* $y = x - 4$ *c.* $y = 2x + 5$
d. $x + y = 6$ *e.* $x - y = 3$ *f.* $3x - 2y = 12$

7. Draw the graph of each of the following lines using its slope and y-intercept:
a. $y = 2x + 5$ *b.* $y = 3x - 1$ *c.* $y = 4x$
d. $y = \frac{2}{3}x + 1$ *e.* $y = -\frac{4}{3}x + 2$ *f.* $3x + y = 4$
g. $y - 2x = 8$ *h.* $3x + y = 4$ *i.* $3x + 4y = 12$

8. In each part, state whether or not the two lines are parallel.
a. $y = 3x + 2, y = 3x + 5$ *b.* $y = -2x - 6, y = -3x - 6$
c. $y - 2x = 8, 2x - y = 4$ *d.* $2x + 3y = 12, 3x + 2y = 8$

9. Write an equation of the line whose slope is 4 and which passes through a point on the y-axis 2 units above the origin.

10. Write an equation of the line whose slope is 3 and which passes through a point on the y-axis 5 units below the origin.

11. Write an equation of the line which is:
a. parallel to the line $y = 3x - 5$ and whose y-intercept is 7.
b. parallel to the line $y - 2x = 4$ and whose y-intercept is -1.
c. parallel to the line $2x + 3y = 6$ and which passes through the origin.

12. Write an equation of the line which passes through the point $(3, 2)$ and is parallel to the line $y = 5x + 1$.

13. Write an equation of the line which passes through the point $(1, 6)$ and is parallel to the line $y - 3x = 5$.

14. Write an equation of the line which passes through the point $(-2, 1)$ and is parallel to the line $2y - 4x = 9$.

15. Write an equation of the line which is parallel to the line $y = 3x + 5$ and which has the same y-intercept as the line $y = 4x - 3$.

16. Write an equation of the line which is perpendicular to the line $y = \frac{2}{3}x$ and which passes through the origin.

17. Write an equation of the line which is perpendicular to the line $2x + y = 3$ and whose y-intercept is 4.

18. Write an equation of the line which passes through the point $(2, 4)$ and which is perpendicular to the line $y = -\frac{2}{3}x + 2$.

19. Tell whether each of the following statements is true or false: The straight line whose equation is $y = 2x$ (*a*) does not pass through the origin. (*b*) has each abscissa twice its ordinate. (*c*) is parallel to the line which passes through the points $(1, -3)$ and $(3, 1)$.

20. (*a*) Using graph paper, on the same set of coordinate axes draw the graph of each of the following equations: (1) $y = 6$ (2) $y = 3x$ (3) $y = \frac{3}{2}x + 6$. (*b*) Find the coordinates of the vertices of the triangle whose sides are the line segments joining the points of intersection of the graphs made in answer to *a*.

21. Given points $(-1, -2)$ and $(3, 4)$.

a. Find the distance between the two points.
b. Find the slope of the line containing these points.
c. Find the coordinates of the midpoint of the segment determined by these points.
d. Write an equation of the line containing the two given points.
e. If the point $(2, k)$ lies on the line determined in part *d*, find the value of k.

22. Given points $A(3, 1)$, $B(0, -1)$, and $C(-3, -3)$.

a. Write an equation of the line which passes through point A and is parallel to the y-axis.
b. Write an equation of the line which passes through point B and has a slope of 1.
c. Show that A, B, and C lie on the same straight line.
d. Write an equation of the line which is parallel to $\overleftrightarrow{AB}$ and passes through the origin.

23. In isosceles triangle ABC with vertices $A(3, -1)$, $B(7, 3)$, and $C(-1, 7)$, $\overline{CD}$ is the altitude to $\overline{AB}$.

a. If the slope of $\overline{CD}$ is -1, write an equation of the line passing through C and D.
b. Write an equation of the line passing through A and B.
c. Using coordinate geometry, show that the altitude of isosceles $\triangle ABC$ intersects the base $\overline{AB}$ at its midpoint.

24. The vertices of parallelogram $ABCD$ are $A(-2, 4)$, $B(2, 6)$, $C(7, 2)$, and $D(k, 0)$.

a. Find the slope of line $\overleftrightarrow{AB}$.
b. Express the slope of line $\overleftrightarrow{CD}$ in terms of k.
c. Using the results found in answer to *a* and *b*, find the value of k.
d. Write an equation of line $\overleftrightarrow{BD}$.

25. In triangle ABC, side $\overline{AB}$ lies on the line whose equation is $y = x + 3$ and side $\overline{AC}$ lies on the y-axis. The coordinates of M and N, the midpoints of sides $\overline{AC}$ and $\overline{BC}$, are $(0, -3)$ and $(2, k)$ respectively.

a. Find the coordinates of A and of C.

b. Find the slope of $\overline{AB}$.
c. Express the slope of $\overline{MN}$ in terms of k.
d. Find the value of k.
e. Write an equation of the line through M and N.

26. Given the points $A(x, 4)$, $B(5, 6)$, $C(7, 5)$, and $D(11, 6)$.
a. Find the slope of the line through the points C and D.
b. Write an expression which represents the slope of the line through the points A and B.
c. Find the value of x that will make the line through A and B parallel to the line through C and D.
d. Write an equation of the line passing through the point B and perpendicular to the y-axis.
e. Write an equation of the line passing through the origin and point D.

27. The points $A(-2, 0)$, $B(10, 3)$, $C(5, 7)$, and $D(2, k)$ are the vertices of a trapezoid whose bases are $\overline{AB}$ and $\overline{DC}$.
a. Find the slope of $\overleftrightarrow{AB}$.
b. Express the slope of $\overleftrightarrow{DC}$ in terms of k.
c. Using the results found in answer to parts *a* and *b*, find the value of k.
d. Show by means of slopes that $\overleftrightarrow{AB}$ does not pass through the origin.
e. Write an equation of $\overleftrightarrow{AB}$.

28. The vertices of quadrilateral $ABCD$ are $A(-6, -6)$, $B(14, 4)$, $C(3, 5)$, and $D(-5, 1)$.
a. Show by means of slopes that $\overline{DC}$ is parallel to $\overline{AB}$.
b. Write an equation of line $\overleftrightarrow{BC}$.

29. *a*. On a set of coordinate axes, plot the points $A(1, 1)$, $B(10, 4)$, $C(7, 7)$, $D(7, 3)$, and $E(5, 5)$.
b. Using the formula for the slope of a line, show that A, E, and C lie on the same straight line.
c. Point D lies on $\overline{AB}$. Show that triangle ADE is similar to triangle ABC.

30. Given the points $A(2, 4)$, $B(6, 13)$, and $C(x, y)$.
a. Write an equation of the line through A and C if its slope is -1.
b. Write an equation of the line through B and C if its slope is $\frac{1}{2}$.
c. Find the coordinates of point C.

31. The vertices of parallelogram $ABCD$ are $A(1, 1)$, $B(4, 6)$, $C(x, y)$, and $D(2, 10)$.
a. Express in terms of x and y the slopes of lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{DC}$.
b. Using the results obtained in part *a*, write two equations that can be used to find x and y.
c. Find the coordinates of C.

32. The vertices of an isosceles triangle ABC are $A(0, 0)$, $B(2r, 2s)$, and $C(2s, 2r)$, where r and s are positive and unequal.

a. Find in terms of r and s the coordinates of D, the midpoint of base $\overline{BC}$.

b. Find the numerical value of the slope of $\overline{AD}$.

c. Find the numerical value of the slope of $\overline{BC}$.

d. Write an equation of the line which contains points A and D.

e. Point P is a point on the median $\overline{AD}$. If its abscissa is k, find its ordinate.

33. In the figure, $\overline{AM}$ and $\overline{BN}$ are medians of $\triangle ABC$.

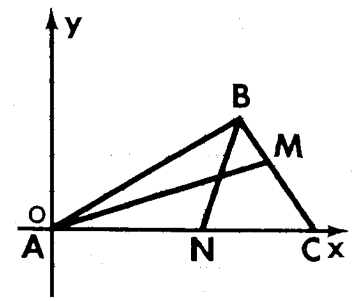
a. The coordinates of A, B, and C are $(0, 0)$, $(6k, 4k)$, and $(8k, 0)$ respectively.

(1) Find the coordinates of point M in terms of k.

(2) Find the numerical value of the slope of $\overleftrightarrow{AM}$.

(3) Write an equation of $\overleftrightarrow{AM}$.

b. If $k = 3$, write an equation of $\overleftrightarrow{BN}$.



Ex. 33

10. Locus in Coordinate Geometry

We have learned that a locus may be a point, a straight line, a collection of several lines, or a curved line that is the set of all points, and only those points, that satisfy a given set of conditions. When the coordinates of all points on a locus determined by a given set of conditions satisfy an equation, and when all points not on the locus do not satisfy the equation, such an equation is called the ***equation of the locus.*** The equation of a locus gives the relationship which exists between the coordinates x and y of all points on the locus.

In Fig. 11–28, we see a set of points lying on a straight line, the abscissa of each point being the number 3. If we take any point P, for example $(-2, 3)$, which is not on this line, its coordinates do not satisfy the equation of the line $(x = 3)$. We see, therefore, that the locus of points whose abscissas are the constant 3 is a line parallel to the y-axis and 3 units to the right of it. The equation of this line is $x = 3$.

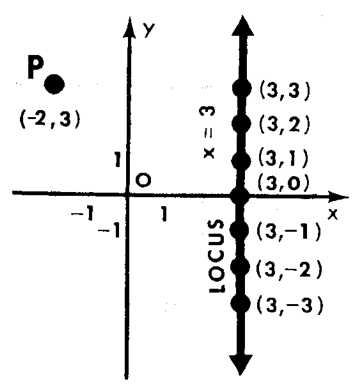


Fig. 11–28

Theorem 159. **The locus of points whose abscissas are the constant a is a line which is parallel to the y-axis and whose equation is $x = a$.**

In Fig. 11–29, we see a set of points lying on a straight line, the ordinate of each point being the number -2. If we take any point P, for example (2, 1), which is not on this line, its coordinates do not satisfy the equation of the line ($y = -2$). We see, therefore, that the locus of points whose ordinates are the constant -2 is a line parallel to the x-axis and 2 units below it. The equation of this line is $y = -2$.

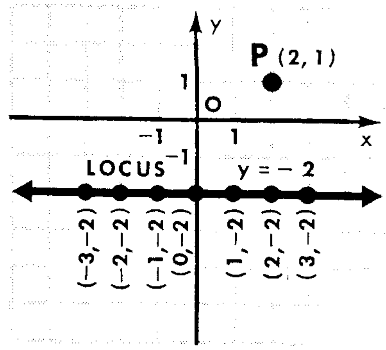


Fig. 11–29

***Theorem 160.* The locus of points whose ordinates are the constant *b* is a line which is parallel to the *x*-axis and whose equation is $y = b$.**

In Fig. 11–30, we see a set of points lying on a straight line, the ordinate of each point being 2 times the abscissa of that point. If we take any point P, for example (2, -3), which is not on this line, its coordinates do not satisfy the equation of the line ($y = 2x$). We see, therefore, that the locus of points in which the ordinate of each point is twice the abscissa of that point is a line which passes through the origin and whose slope is 2. The equation of this line is $y = 2x$.

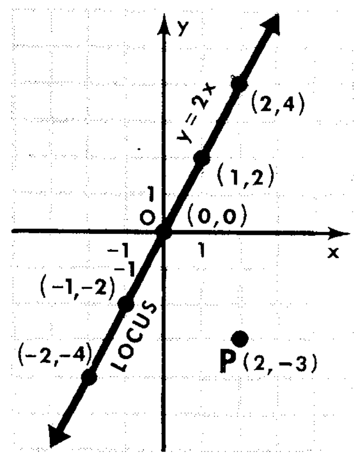


Fig. 11–30

***Theorem 161.* The locus of points in which the ordinate of each point is *m* times the abscissa of that point is a line which passes through the origin and whose equation is $y = mx$.**

In Fig. 11–31, we see a set of points lying on a straight line, the ordinate of each point being the sum of 3 times the abscissa of that point and (-2). If we take any point P, for example (1, -4), which is not on this line, its coordinates do not satisfy the equation of the line, which is $y = 3x + (-2)$. We see, therefore, that the locus of points in which the ordinate of each point is the sum of 3 times the abscissa of that point and (-2) is a line whose slope is 3 and which intersects the y-axis 2 units below the origin. The equation of this line can be written $y = 3x + (-2)$, or, more simply, $y = 3x - 2$.

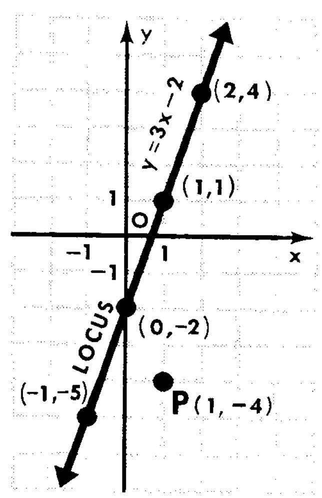


Fig. 11–31

Theorem 162. **The locus of points in which the ordinate of each point is the sum of *m* times the abscissa of that point and the number *b* is a line whose slope is *m*, whose *y*-intercept is *b*, and whose equation is $y = mx + b$.**

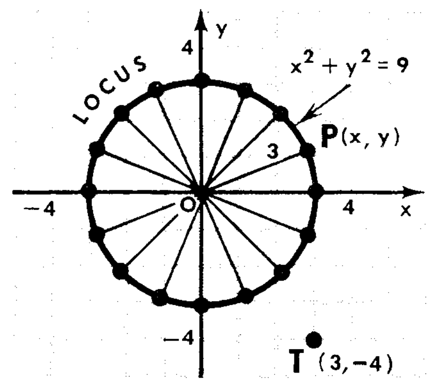


Fig. 11–32

In Fig. 11–32, we see a set of points lying on a circle, each point being 3 units from a fixed point, the origin. If we take any point T, for example $(3, -4)$, which is not on this circle, point T is not 3 units from the origin, and its coordinates do not satisfy the equation of the circle ($x^2 + y^2 = 9$). We see, therefore, that the locus of points 3 units from the origin is a circle whose center is at the origin and whose radius is 3. If $P(x, y)$ is any point on this circle, by using either the distance formula or the Pythagorean Theorem, we find that the equation of the circle can be written $x^2 + y^2 = (3)^2$, or $x^2 + y^2 = 9$.

Theorem 163. **The locus of points whose distance from the origin is *r* is a circle whose center is the origin, the length of whose radius is *r*, and whose equation is $x^2 + y^2 = r^2$.**

MODEL PROBLEMS

1. Write an equation of the locus of points in which the ordinate of each point is 3 more than 4 times the abscissa of that point.

Solution:

1. Let $P(x, y)$ represent the points on the locus that satisfy the given condition: The ordinate of each point is 3 more than 4 times the abscissa of that point.
2. If we replace "ordinate" by "y" and "abscissa" by "x," we obtain the equation $y = 4x + 3$.

Answer: $y = 4x + 3$.

2. Write an equation of the locus of points equidistant from the points (2, 2), and (6, 2).

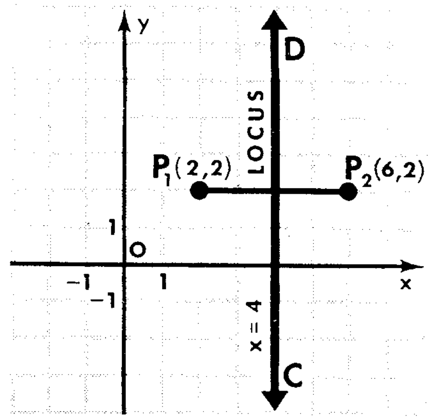
Solution:

1. The locus of points equidistant from the given points $P_1(2, 2)$ and $P_2(6, 2)$ is $\overleftrightarrow{CD}$, the perpendicular bisector of the line segment $\overline{P_1P_2}$.

2. $\overleftrightarrow{CD}$ is parallel to the y-axis and 4 units to the right of the y-axis.

3. Therefore, the equation of $\overleftrightarrow{CD}$ is $x = 4$.

Answer: $x = 4$.



3. *a.* Write an equation of the locus of points whose distance from the origin is 5.

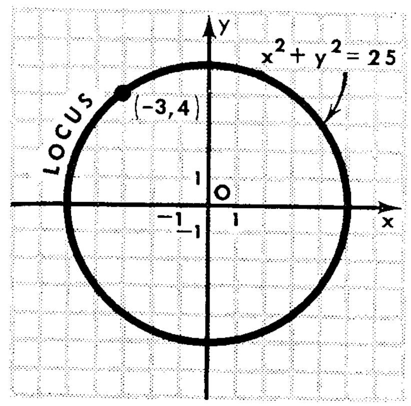
b. Determine whether the point $(-3, 4)$ is on the locus.

Solution:

(*a*) 1. The locus of points whose distance from the origin is 5 is a circle whose center is O and whose radius is 5.

2. The equation of this circle is $x^2 + y^2 = (5)^2$, or $x^2 + y^2 = 25$.

Answer: $x^2 + y^2 = 25$.



(*b*) 1. If the point $(-3, 4)$ is to be on the locus, $x = -3$, $y = 4$ must satisfy the equation $x^2 + y^2 = 25$.

2. $(-3)^2 + (4)^2 \stackrel{?}{=} 25$

3. $9 + 16 \stackrel{?}{=} 25$

4. $25 = 25$

Answer: The point $(-3, 4)$ is on the locus.

EXERCISES

1. Write an equation and plot the graph of the equation of the line which is the locus of all points:

a. whose ordinate is 5.
b. whose ordinate is 8.
c. whose abscissa is 4.
d. whose abscissa is 0.
e. whose ordinate is -1.
f. whose abscissa is -6.
g. whose ordinate is 4 times its abscissa.
h. whose ordinate is equal to its abscissa.
i. whose ordinate is $\frac{3}{4}$ of its abscissa.
j. whose ordinate is 5 more than its abscissa.
k. whose ordinate is 4 less than its abscissa.
l. whose ordinate exceeds its abscissa by 3.
m. whose ordinate is 2 greater than 3 times its abscissa.
n. whose ordinate is 1 less than 4 times its abscissa.
o. whose abscissa is 2 more than its ordinate.
p. whose abscissa is 3 less than twice its ordinate.
q. the sum of whose abscissa and ordinate is 10.
r. the sum of whose coordinates is 6.
s. the sum of whose ordinate and abscissa is 8.
t. whose ordinate decreased by its abscissa is 5.
u. whose abscissa decreased by its ordinate is 4.
v. whose ordinate is 3 more than one-half its abscissa.
w. whose abscissa is 4 less than one-third its ordinate.

2. Write an equation of the locus of all points equidistant from the points:

a. (0, 2) and (0, 10)
b. (3, 0) and (9, 0)
c. (2, 6) and (2, 12)
d. (9, 3) and (15, 3)
e. (0, -4) and (0, 8)
f. (-3, 0) and (5, 0)
g. (-1, 4) and (-1, 12)
h. (1, -5) and (7, -5)

3. Choose the correct answer: The equation $x = 3$ represents the locus of points equidistant from the two points (*a*) (2, 0) and (1, 0) (*b*) (0, 3) and (3, 0) (*c*) (10, 0) and (-4, 0).

4. Write an equation of the locus of points:

a. whose abscissa is 6.
b. whose abscissa is 9.
c. whose abscissa is -2.
d. whose abscissa is -4.
e. whose ordinate is 1.
f. whose ordinate is 4.
g. whose ordinate is -5.
h. whose ordinate is -8.

5. Write an equation of the locus of all points:

a. 2 units from the x-axis and above it.
b. 5 units from the y-axis and to the right of it.
c. 4 units from the y-axis and to the left of it.

d. 3 units from the x-axis and below it.
e. equidistant from the x-axis and y-axis and whose coordinates have the same sign.
f. equidistant from the x-axis and y-axis and whose coordinates have opposite signs.

6. Write an equation of the locus of points which are 6 units above the x-axis.

7. Write an equation of the locus of points which are 5 units to the right of the y-axis.

8. Write an equation of the locus of points which are 12 units below the x-axis.

9. Write an equation of the straight line passing through the point (3, 4) and perpendicular to the x-axis.

10. Write an equation of the straight line passing through the point $(-1, 3)$ and parallel to the y-axis.

11. Write an equation of the straight line passing through the point (5, 2) and perpendicular to the y-axis.

12. Write an equation of the line passing through the point $(-2, -4)$ and parallel to the x-axis.

13. Write an equation of the locus of points which are 3 units from the line whose equation is $y = 7$.

14. Choose the correct answer: The locus of points in the coordinate plane at a distance of 8 units from the x-axis consists of the graph(s) of the equation(s) (*a*) $y = 8$ (*b*) $x = 8$ (*c*) $y = 8, y = -8$ (*d*) $x = 8, x = -8$.

15. Write an equation of the locus of points which are 5 units from the line whose equation is $y = -2$.

16. Write an equation of the locus of points 2 units from the line whose equation is $x = 6$.

17. Write an equation of the locus of points which are 3 units from the line whose equation is $x = -1$.

18. Give the coordinates of a point on the x-axis which is equidistant from the points (8, 5) and (12, 5).

19. Give the coordinates of a point on the y-axis which is equidistant from the points (4, 3) and $(4, -9)$.

20. Write an equation of the locus of the centers of circles which are tangent to both lines:
a. $y = 10$ and $y = 4$ *b.* $x = 2$ and $x = 12$
c. $y = -4$ and $y = -6$ *d.* $x = -8$ and $x = -2$
e. $y = 6$ and $y = -14$ *f.* $x = -5$ and $x = 9$

21. Write an equation of the locus of all points whose distance from the origin is:
a. 4 units *b.* 9 units *c.* 12 units *d.* 6 units *e.* $2\frac{1}{2}$ units

22. Describe the locus whose algebraic representation is:
a. $x^2 + y^2 = 36$ *b.* $x^2 + y^2 = 64$ *c.* $x^2 + y^2 = \frac{81}{4}$ *d.* $x^2 + y^2 = 3$

23. Determine whether the given point is on the locus whose equation is given.
a. $x + y = 10$, (8, 2) *b.* $x - y = 4$, (8, −4)
c. $2y + x = 7$, (1, 3) *d.* $4x - 2y = 10$, (3, −1)
e. $y = 3$, (2, 3) *f.* $x = -8$, (3, −8)

24. Find the value of k so that the graph of the given equation will pass through the given point.
a. $x + y = k$, (2, 3) *b.* $2x - 3y = k$, (1, −4)

25. A point is on the locus whose equation is given. Find its ordinate if its abscissa is the number indicated.
a. $x + 3y = 5$, (2) *b.* $2x - y = 6$, (−1) *c.* $3x + 2y = 8$, (6)

26. A point is on the locus whose equation is given. Find its abscissa if its ordinate is the number indicated.
a. $x - y = 3$, (6) *b.* $3x - y = 10$, (−2) *c.* $3x + 2y = 5$, (−5)

27. (*a*) Write an equation of the locus of points whose distance from the origin is 13. (*b*) Determine whether the point (−12, 5) is on the locus. (*c*) Determine whether the point (8, −9) is on the locus.

28. (*a*) Write an equation of the locus of points whose distance from the origin is 5. (*b*) Without constructing the circle, determine the coordinates of every point on the locus whose abscissa is 3.

29. Line $\overleftrightarrow{CD}$ passes through the points (2, 0) and (8, 0). Point $P(4, 0)$ is on line $\overleftrightarrow{CD}$. Write an equation of the locus of the centers of circles which are tangent to line $\overleftrightarrow{CD}$ at point P.

30. Write an equation of the locus of the centers of circles which will pass through the two given points $A(0, 1)$ and $B(0, 5)$.

31. The vertices of triangle ABC are $A(5, 2)$, $B(10, 2)$, and $C(7, 8)$. Write equations for the locus of points which are vertices of triangles equal in area to triangle ABC and which have $\overline{AB}$ as their base.

32. The endpoints of line segment $\overline{AB}$ are (−4, 0) and (4, 0). Write an equation of the locus of points which are the vertices C of all right triangles ABC having $\overline{AB}$ as their hypotenuse.

33. Write an equation of the locus of points equidistant from the circles whose equations are $x^2 + y^2 = 4$ and $x^2 + y^2 = 64$.

11. Intersection of Loci in Coordinate Geometry

If points are to satisfy each of two conditions, these points must be members of the intersection set of the set of points that satisfy the first condition and the set of points that satisfy the second condition. In coordinate geometry, these points can be located graphically or geometrically by using the following procedure:

1. Draw the graph of the locus of points that satisfy the first condition.
2. Draw the graph of the locus of points that satisfy the second condition.
3. Locate the points of intersection of these loci. These points must be the required points because they satisfy both conditions.

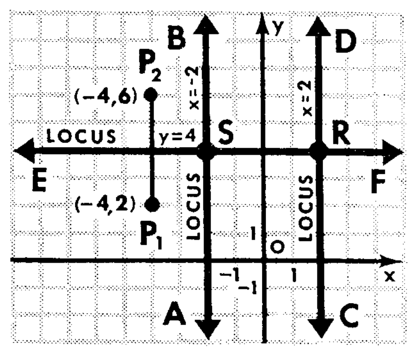
MODEL PROBLEMS

1. *a.* Describe the locus of points 2 units from the y-axis and write an equation of this locus.

b. Describe the locus of points equidistant from the points $P_1(-4, 2)$ and $P_2(-4, 6)$ and write an equation of this locus.

c. Find the number of points which satisfy both conditions stated in part a and part b and give the coordinates of each point.

Solution:



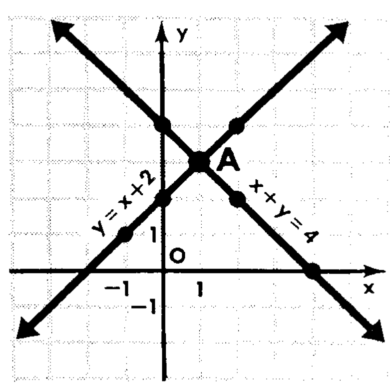
a. The locus is the union of a pair of lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, parallel to the y-axis, each line being 2 units from the y-axis. An equation of $\overleftrightarrow{CD}$ is $x = 2$. An equation of $\overleftrightarrow{AB}$ is $x = -2$.

b. The locus is $\overleftrightarrow{EF}$, the perpendicular bisector of the line segment $\overline{P_1P_2}$. An equation of $\overleftrightarrow{EF}$ is $y = 4$.

c. There are two points, R and S, which are the points of intersection of the loci and which therefore satisfy both stated conditions. The coordinates of R are $(2, 4)$. The coordinates of S are $(-2, 4)$.

2. Find graphically the common solution for the system of equations
$$\begin{cases} x + y = 4 \\ y = x + 2 \end{cases}$$

Solution:



1. The locus of points whose coordinates satisfy the equation $x + y = 4$ is a straight line. Graph this line.

If $x = \ldots,$	0	2	4
then $y = \ldots.$	4	2	0

[The solution continues on the next page.]

2. The locus of points whose coordinates satisfy the equation $y=x+2$ is a straight line. Graph this line.

If $x = \ldots,$	-1	0	2
then $y = \ldots.$	1	2	4

3. Point $A(1, 3)$ is the point of intersection of the two graphs that were drawn. Hence, $\{(1, 3)\}$ is the intersection of the solution sets of the given equations. Therefore, $x = 1, y = 3$ is the common solution for the given system of equations.

Answer: $x = 1, y = 3$, or $(1, 3)$.

NOTE. Recall that when you studied algebra, you learned algebraic methods of discovering that the common solution for the given system of equations is the ordered pair of numbers $(1, 3)$.

EXERCISES

1. (*a*) Represent graphically the locus of points (1) 3 units from the line $x = 1$. (2) 4 units from the line $y = -2$. (*b*) Write the equations for the loci represented in part *a*. (*c*) Find the coordinates of the points of intersection of the loci represented in answer to part *a*.

2. *a.* Draw the locus of points equidistant from the points $(4, 1)$ and $(4, 5)$ and write an equation for this locus.
b. Draw the locus of points equidistant from the points $(3, 2)$ and $(9, 2)$ and write an equation for this locus.
c. Find the number of points that satisfy both conditions stated in part *a* and part *b*. Give the coordinates of each point found.

3. The total number of points equidistant from both the x-axis and the y-axis and 3 inches distant from the origin is (*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4.

4. (*a*) On a sheet of graph paper, draw two perpendicular axes and on this set of axes represent graphically the locus of points: (1) 8 units from the y-axis (2) 10 units from the origin. (*b*) Write equations for the loci represented in part *a*. (*c*) Find the coordinates of the points of intersection of the loci represented in answer to part *a*.

5. (*a*) On a sheet of graph paper, draw the graph of $y = 6$ and $y = 2$. (1) Draw the graph of the locus of the centers of circles which are tangent to both lines whose equations are $y = 6$ and $y = 2$. (2) Write an equation of this locus. (*b*) Using the same set of axes, draw the graph of the lines whose equations are $x = 4$ and $x = 8$. (1) Draw the graph of the locus of the centers of circles which are tangent to both lines whose equations are $x = 4$ and $x = 8$. (2) Write an equation of this locus. (*c*) What are the coordinates of the center of a circle which will be tangent to the lines whose equations are $y = 2$, $y = 6$, $x = 4$, and $x = 8$? (*d*) What is the length of a radius of this circle?

6. The vertices of triangle RST are $R(2, 3)$, $S(6, 3)$, and $T(3, 10)$. (*a*) Draw the graph and write an equation of the locus of points equidistant from R and S. (*b*) Draw line $\overleftrightarrow{TW}$ which passes through T and is parallel to $\overleftrightarrow{RS}$ and write an equation of $\overleftrightarrow{TW}$. (*c*) Write the coordinates of the points of intersection of the lines drawn in parts *a* and *b*. (*d*) What are the coordinates of a point on line $\overleftrightarrow{TW}$ which can be used as the center of a circle which passes through points R and S?

7. Draw the graphs of the lines whose equations are given and use the graphs to find the coordinates of the point of intersection of:

a. $y = 3x - 1$ and $y = x + 9$ *b.* $y = 5x + 2$ and $y = -2x + 16$

8. Solve the following systems of equations graphically:

a. $x + y = 8$	*b.* $x - y = 2$	*c.* $y = 2x + 1$
$x - y = 4$	$y = 2x$	$x + y = 10$
d. $2x + y = 8$	*e.* $y - x = -2$	*f.* $x - 3y = 9$
$y - x = 2$	$x - 2y = 4$	$2x - y = 8$

9. *a.* Draw the locus of points equidistant from the circles whose equations are $x^2 + y^2 = 4$ and $x^2 + y^2 = 36$. Write an equation of the locus.

b. Draw the locus of points 4 units from the x-axis. Write an equation of the locus.

c. Find the number of points that satisfy both conditions stated in part *a* and part *b*. Write the coordinates of each of the points found.

10. *a.* Using graph paper, draw the triangle whose vertices are $A(1, 2)$, $B(7, 2)$, and $C(5, 6)$.

b. Draw the locus of points equidistant from A and B.

c. Write an equation of the line drawn in answer to part *b*.

d. Write an equation of the line through A and C.

e. Point P in line $\overleftrightarrow{AC}$ is equidistant from points A and B. Find the coordinates of point P.

11. The vertices of a triangle are $A(2, 3)$, $B(8, 3)$, and $C(4, 7)$.

a. Find the equations of the three medians of triangle ABC.

b. Show that the three medians pass through the same point; that is, they are *concurrent*.

12. The vertices of triangle ABC are $A(1, 1)$, $B(7, 11)$, and $C(7, 5)$. Prove that the medians of triangle ABC are concurrent.

13. The vertices of triangle ABC are $A(0, 0)$, $B(6, 0)$, and $C(0, 8)$. Prove that the perpendicular bisectors of the sides of triangle ABC are concurrent.

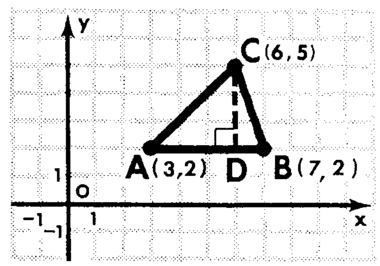
12. Areas in Coordinate Geometry

To find areas of polygons in coordinate geometry, we use the area formulas previously developed. When a figure has one or more sides parallel to either of the axes, the process of finding its area usually becomes simplified.

MODEL PROBLEMS

1. Find the area of a triangle whose vertices are $A(3, 2)$, $B(7, 2)$, and $C(6, 5)$.

Solution:



1. Since points A and B have the same ordinate, $\overline{AB}$ is parallel to the x-axis.
2. In $\triangle ABC$, $\overline{AB}$ is the base and $AB = 4$.
3. Draw altitude $\overline{CD}$. $CD = 3$.
4. Area of $\triangle ABC = \frac{1}{2}AB \times CD$.
5. Area of $\triangle ABC = \frac{1}{2} \times 4 \times 3 = 6$.

Answer: Area = 6 square units.

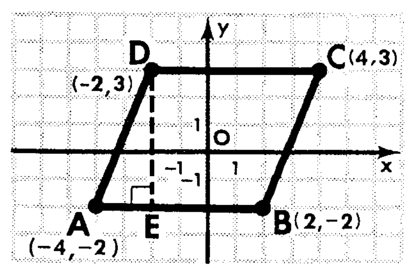
2. Points $A(-4, -2)$, $B(2, -2)$, $C(4, 3)$, and $D(-2, 3)$ are the vertices of quadrilateral $ABCD$.

a. Plot these points on graph paper and draw the quadrilateral.

b. What kind of quadrilateral is $ABCD$?

c. Find the area of quadrilateral $ABCD$.

Solution:



a. See graph.

b. 1. $\overline{AB}$ is parallel to $\overline{CD}$.

2. $AB = |2 - (-4)| = |2 + 4| = |6| = 6$ and $CD = |4 - (-2)| = |4 + 2| = |6| = 6$. Hence, $AB = CD$, or $\overline{AB} \cong \overline{CD}$.

3. $ABCD$ is a parallelogram since it is a quadrilateral with one pair of opposite sides both parallel and congruent.

c. 1. Draw altitude $\overline{DE}$ to side $\overline{AB}$.
$DE = 5$.

2. Area of $\square ABCD = AB \times DE$.

3. Area of $\square ABCD = 6 \times 5 = 30$.

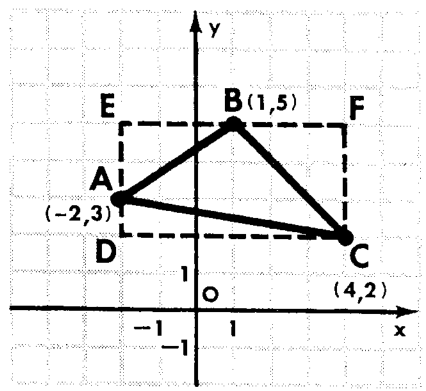
Answer: Area = 30 square units.

Often the polygon whose area we are trying to find does not have sides parallel to either of the axes. We then draw lines parallel to the axes through the vertices, forming rectangles and trapezoids whose areas can be used to find the area of the given polygon, as shown in Model Problems 3 and 4 below.

3. Plot the points $A(-2, 3)$, $B(1, 5)$, $C(4, 2)$ and find the area of $\triangle ABC$.

Method 1

Plan: Through points B and C, draw lines parallel to the x-axis. Through points A and C, draw lines parallel to the y-axis, forming rectangle $CDEF$. The area of $\triangle ABC$ is equal to the area of rectangle $CDEF$ minus the sum of the areas of right triangles CDA, BEA, and BFC.



Solution:

1. Area of $\triangle ABC$ = area of rectangle $CDEF$ – (area of $\triangle CDA$ + area of $\triangle BEA$ + area of $\triangle BFC$).

2. Area of rectangle $CDEF = DC \times DE = 6 \times 3 = 18$.

3. Area of rt. $\triangle CDA = \frac{1}{2}DC \times DA = \frac{1}{2}(6)(1) = 3$.

4. Area of rt. $\triangle BEA = \frac{1}{2}BE \times EA = \frac{1}{2}(3)(2) = 3$.

5. Area of rt. $\triangle BFC = \frac{1}{2}BF \times FC = \frac{1}{2}(3)(3) = 4.5$.

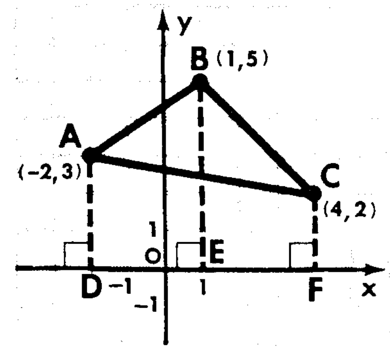
6. Area of $\triangle ABC = 18 - (3 + 3 + 4.5) = 18 - 10.5$.

7. Area of $\triangle ABC = 7.5$.

Answer: Area of $\triangle ABC$ = 7.5 square units.

Method 2

Plan: Draw $\overleftrightarrow{AD} \parallel$ the y-axis, $\overleftrightarrow{BE} \parallel$ the y-axis, $\overleftrightarrow{CF} \parallel$ the y-axis, forming trapezoids $DEBA$, $EFCB$, and $DFCA$. The area of $\triangle ABC$ can be found by adding the areas of trapezoids $DEBA$ and $EFCB$ and then subtracting the area of trapezoid $DFCA$ from the sum.



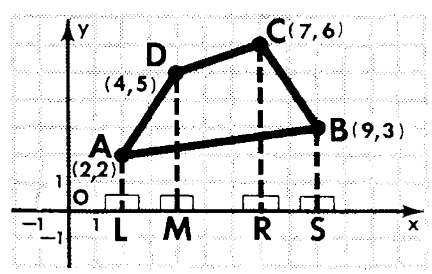
Solution:

1. Area of $\triangle ABC$ = (area of trapezoid $DEBA$ + area of trapezoid $EFCB$) − area of trapezoid $DFCA$.
2. Area of trapezoid $DEBA = \frac{1}{2}(DE)(DA + EB) = \frac{1}{2}(3)(3 + 5) = \frac{1}{2}(3)(8) = 12$.
3. Area of trapezoid $EFCB = \frac{1}{2}(EF)(EB + FC) = \frac{1}{2}(3)(5 + 2) = \frac{1}{2}(3)(7) = 10.5$.
4. Area of trapezoid $DFCA = \frac{1}{2}(DF)(DA + FC) = \frac{1}{2}(6)(3 + 2) = \frac{1}{2}(6)(5) = 15$.
5. Area of $\triangle ABC = (12 + 10.5) - 15 = 22.5 - 15$.
6. Area of $\triangle ABC = 7.5$.

Answer: Area of $\triangle ABC$ = 7.5 square units.

4. Find the area of the polygon whose vertices are $A(2, 2)$, $B(9, 3)$, $C(7, 6)$, and $D(4, 5)$.

Plan: Draw $\overleftrightarrow{AL} \parallel$ the y-axis, $\overleftrightarrow{DM} \parallel$ the y-axis, $\overleftrightarrow{CR} \parallel$ the y-axis, $\overleftrightarrow{BS} \parallel$ the y-axis, forming trapezoids $LMDA$, $MRCD$, $RSBC$, and $LSBA$. The area of polygon $ABCD$ can be found by adding the areas of trapezoids $LMDA$, $MRCD$, $RSBC$ and subtracting the area of trapezoid $LSBA$ from the sum.



Solution:

1. Area of $ABCD$ = (area of trapezoid $LMDA$ + area of trapezoid $MRCD$ + area of trapezoid $RSBC$) − area of trapezoid $LSBA$.

2. Area of trapezoid $LMDA = \frac{1}{2}(LM)(AL + DM) = \frac{1}{2}(2)(2+5) = \frac{1}{2}(2)(7) = 7$.

3. Area of trapezoid $MRCD = \frac{1}{2}(MR)(DM + CR) = \frac{1}{2}(3)(5+6) = \frac{1}{2}(3)(11) = 16.5$.

4. Area of trapezoid $RSBC = \frac{1}{2}(RS)(CR + BS) = \frac{1}{2}(2)(6+3) = \frac{1}{2}(2)(9) = 9$.

5. Area of trapezoid $LSBA = \frac{1}{2}(LS)(AL + BS) = \frac{1}{2}(7)(2+3) = \frac{1}{2}(7)(5) = 17.5$.

6. Area of $ABCD = (7 + 16.5 + 9) - 17.5 = 32.5 - 17.5$.

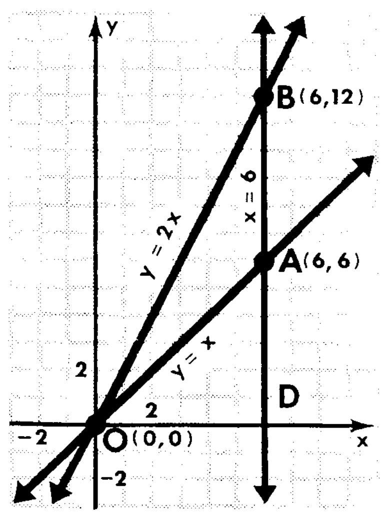
7. Area of $ABCD = 15$.

Answer: Area of $ABCD$ = 15 square units.

5. *a.* Using graph paper, on the same set of axes draw the graph of each of the following equations: (1) $x = 6$ (2) $y = x$ (3) $y = 2x$.

b. Find the coordinates of the vertices of the triangle whose sides are the line segments joining the points of intersection of the graphs made in answer to part *a*.

c. Find the area of the triangle described in part *b*.



Solution:

a. See graph.

b. 1. Line $y = x$ intersects line $y = 2x$ at $O(0, 0)$.

2. Line $y = x$ intersects line $x = 6$ at $A(6, 6)$.

3. Line $y = 2x$ intersects line $x = 6$ at $B(6, 12)$.

Answer: $O(0, 0)$, $A(6, 6)$, and $B(6, 12)$.

c. 1. Area of $\triangle AOB = \frac{1}{2}$ the length of side $\overline{BA} \times$ the length of altitude $\overline{OD}$ drawn to side $\overline{BA}$ extended.

2. Area of $\triangle AOB = \frac{1}{2} BA \times OD = \frac{1}{2}(6)(6)$.

3. Area of $\triangle AOB = 18$.

Answer: Area of $\triangle AOB$ = 18 square units.

EXERCISES

1. Find the area of a rectangle whose vertices are:
a. (0, 0), (8, 0), (0, 5), (8, 5) *b.* (−2, 3), (4, 3), (−2, 8), (4, 8)

2. Find the area of a parallelogram whose vertices are:
a. (0, 0), (4, 0), (2, 3), (6, 3) *b.* (−2, 8), (−3, 4), (5, 8), (4, 4)

3. Find the area of a triangle whose vertices are:
a. (0, 0), (12, 0), (2, 8) *b.* (0, 8), (0, −3), (4, 5)
c. (2, −2), (8, −2), (4, −6)

4. Find the area of a right triangle whose vertices are $A(0, 0)$, $B(6, 0)$, and $C(0, 3)$.

5. Find the area of a trapezoid whose vertices are:
a. (0, 0), (12, 0), (2, 6), (7, 6) *b.* (0, 4), (0, −8), (3, 1), (3, −4)

6. Points $Q(8, 2)$, $R(14, 6)$, $S(4, 6)$, $T(-2, 2)$ are the vertices of quadrilateral $QRST$. (*a*) Plot these points on graph paper and draw the quadrilateral. (*b*) What kind of quadrilateral is $QRST$? (*c*) Find the area of quadrilateral $QRST$.

7. Points $C(1, -4)$, $D(9, -4)$, $E(9, 2)$, $F(1, 3)$ are the vertices of quadrilateral $CDEF$. (*a*) Plot these points on graph paper and draw the quadrilateral. (*b*) Find the lengths of $\overline{CD}$, $\overline{DE}$, and diagonal $\overline{CE}$. (*c*) What kind of quadrilateral is $CDEF$? (*d*) Find the area of the quadrilateral.

8. Plot the points $A(2, 4)$, $B(10, 6)$, $C(8, 12)$. Through A, draw lines parallel to both axes. Through B, draw a line parallel to the y-axis. Through C, draw a line parallel to the x-axis. (*a*) Find the area of the rectangle thus formed. (*b*) Draw $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. Find the area of triangle ABC.

9. Plot the points $R(5, -4)$, $S(2, 7)$, $T(-2, 2)$. Through R, draw lines parallel to both axes. Through S, draw a line parallel to the x-axis. Through T, draw a line parallel to the y-axis. (*a*) Find the area of the rectangle thus formed. (*b*) Draw $\overline{RS}$, $\overline{ST}$, and $\overline{TR}$. Find the area of triangle RST.

10. Find the area of a triangle whose vertices are the points:
a. (2, 4), (8, 8), (16, 6) *b.* (6, −2), (8, −10), (12, −6)
c. (6, 4), (9, 2), (13, 6) *d.* (−5, 4), (2, 1), (6, 5)

11. Plot the points $A(2, 5)$, $B(11, 2)$, $C(9, 8)$, and $D(4, 8)$. Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. Find the area of $ABCD$.

12. Find the area of a quadrilateral whose vertices are the points:
a. (−2, 3), (3, 7), (8, 6), (12, 4) *b.* (2, 2), (4, 6), (10, 12), (8, 4)

13. The coordinates of the vertices of triangle RST are $R(4, 5)$, $S(12, 5)$, and $T(8, 11)$. (*a*) Find the length of the altitude of triangle RST drawn to side $\overline{RS}$. (*b*) Find the area of triangle RST.

14. The coordinates of the vertices of triangle ABC are $A(-2, -5)$, $B(7, -2)$, and $C(-2, 1)$. (*a*) Find the length of the altitude of triangle ABC drawn to side $\overline{AC}$. (*b*) Find the area of triangle ABC.

15. Using graph paper, draw triangle RST whose vertices are $R(4, 4)$, $S(12, 10)$, and $T(6, 13)$. (*a*) Find the area of triangle RST. (*b*) Find the length of side $\overline{RS}$. (*c*) Find the length of the altitude drawn to $\overline{RS}$ from T.

16. The vertices of triangle ABC are the points $A(2, 5)$, $B(13, 3)$, and $C(10, 11)$. (*a*) Find the area of triangle ABC. (*b*) Find the length of side $\overline{AC}$. (*c*) Using the results found in answer to parts *a* and *b*, find the length of the altitude drawn from point B to side $\overline{AC}$.

17. The vertices of a triangle are $A(0, 4)$, $B(4, 7)$, and $C(6, 2)$. (*a*) Find the area of triangle ABC. (*b*) Find the length of $\overline{AB}$. (*c*) Find the length of the altitude drawn from C to $\overline{AB}$.

18. The coordinates of the vertices of triangle ABC are $A(4, 4)$, $B(15, 2)$, and $C(12, 8)$. (*a*) Draw triangle ABC on graph paper. (*b*) Show that triangle ABC is a right triangle. (*c*) Find the area of triangle ABC.

19. The vertices of a triangle are $A(0, 6)$, $B(6, 2)$, and $C(12, 10)$. (*a*) Find the area of triangle ABC. (*b*) Find the length of $\overline{BC}$. (*c*) Find the length of the altitude drawn from A to $\overline{BC}$. (*d*) Write an equation of the line determined by point A and the midpoint of $\overline{BC}$.

20. The vertices of a triangle are $A(2, 1)$, $B(5, 8)$, and $C(7, -1)$. (*a*) Using graph paper, draw triangle ABC. (*b*) Find the area of triangle ABC. (*c*) If the median $\overline{AD}$ is drawn to side $\overline{BC}$, find the coordinates of D.

21. In triangle ABC, the coordinates of B are $(-3, -2)$ and those of C are $(5, 4)$. The midpoint of $\overline{AB}$ is M, whose coordinates are $(-3, 2)$. Find the:
a. coordinates of vertex A.
b. coordinates of N, the midpoint of $\overline{BC}$.
c. length of $\overline{MN}$.
d. area of $\triangle MNC$.

22. A triangle whose vertices are $A(2, 3)$, $B(8, 11)$, and $C(0, 7)$ is inscribed in a circle. (*a*) Using the lengths of the sides, show that triangle ABC is a right triangle. (*b*) Find the coordinates of the center of the circumscribed circle. (*c*) Find the area of the circle. [Answer may be expressed in terms of π.]

23. (*a*) Show that the quadrilateral whose vertices are $A(2, 1)$, $B(6, -2)$, $C(10, 1)$, and $D(6, 4)$ is a rhombus. (*b*) Find the area of the rhombus. (*c*) Find a side of the rhombus. (*d*) Find the altitude of the rhombus.

24. (*a*) Show that the quadrilateral whose vertices are the points $A(-7, 3)$, $B(-1, -5)$, $C(5, 3)$, and $D(-1, 11)$ is a rhombus. (*b*) Find the area of the rhombus. (*c*) Find a side of the rhombus. (*d*) Find the altitude of the rhombus.

25. The lines $y = 6$, $y = -2$, $x = 4$, and $x = 10$ intersect. The line segments which join the four points of intersection form a quadrilateral. (*a*) Using graph paper, draw the graphs of these four lines. (*b*) Find the area of the quadrilateral.

26. The lines $x = 8$, $y = 4$, $x = -3$, and $y = -5$ intersect. The line segments which join the four points of intersection form a quadrilateral. (*a*) Using graph paper, draw the graphs of these lines. (*b*) Find the area of the quadrilateral.

27. The lines $x = 0$, $x = 3$, $y = x$, and $y = x + 4$ intersect. The line segments which join the four points of intersection form a quadrilateral. (*a*) Using graph paper, draw the graphs of these lines. (*b*) Find the area of the quadrilateral.

28. The lines $y = x$, $y = x - 5$, $y = 0$, and $y = -3$ intersect. The line segments which join the four points of intersection form a quadrilateral. (*a*) Using graph paper, draw the graphs of these lines. (*b*) Find the area of the quadrilateral.

29. (*a*) Using graph paper, on the same set of axes draw the graph of each of the equations, $y = x$, $y = 3x$, and $y = 9$. (*b*) Find the coordinates of the vertices of the triangle whose sides are the line segments joining the points of intersection of the graphs made in answer to part *a*. (*c*) Find the area of the triangle described in part *b*.

30. (*a*) Using graph paper, on the same set of axes draw the graph of each of the equations, $y = x$, $y = 4x$, and $x = 3$. (*b*) Find the coordinates of the vertices of the triangle whose sides are the line segments joining the points of intersection of the graphs made in answer to part *a*. (*c*) Find the area of the triangle described in part *b*.

13. Line Reflections and Line Symmetry

What Is a Transformation?

In the game of pool, 15 object balls numbered 1 through 15 are placed in a triangular framework called a rack (see Figure A). Let us suppose that the object balls are taken out of the rack, mixed up, and thrown back, as shown in Figure B. Most of the object balls have changed their positions, although a few, such as 2 and 11, remain in their original places.

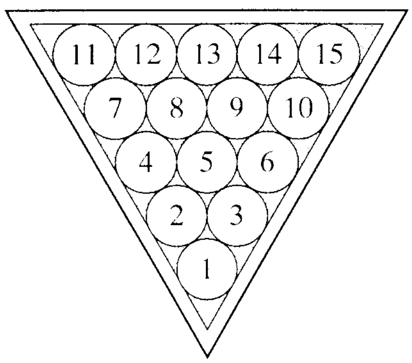


Figure A

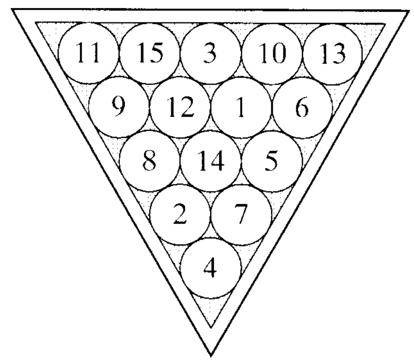


Figure B

The new rack of 15 object balls is merely a *change,* or a ***transformation,*** of the 15 object balls in the original rack.

Imagine that each object ball is like a point, and that the rack containing the object balls is like a plane that contains an infinite number of points. In the same way that object balls in a rack change their positions, points, under a *transformation of the plane,* will move about and change their positions in the plane. At times, some of the points in the plane may remain fixed. After the transformation, or change, takes place, however, the plane must once again appear full and complete, without any missing points, just as the rack of 15 object balls appears full and complete in Figure B.

Comparing the positions of the object balls in the two racks, we find that 1 is replaced by 4, 2 is still 2, 3 is replaced by 7, and so on. The result is a ***one-to-one correspondence*** between the two sets, each of 15 object balls. In other words, each object ball is replaced by one and only one object ball until the rack is again complete.

We will extend this idea to points in a plane. An infinite number of transformations can take place in a plane. In this chapter we will study only a few special transformations.

Line Reflection

It is often possible to see the objects along the shore of a body of water reflected in the water. If a picture of such a scene is folded, the objects can be made to coincide with their images. Each point of the reflection is an

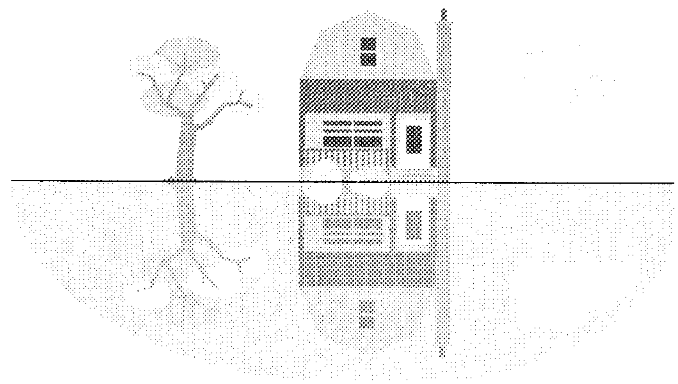
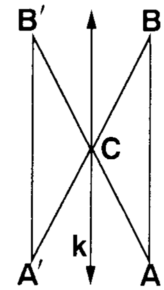
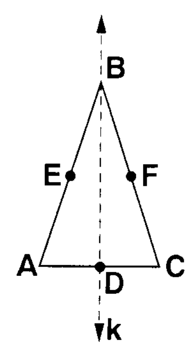


image point of the corresponding point of the object. The line along which the picture is folded is the ***line of reflection,*** and the correspondence between the object points and the image points is called a ***line reflection.*** This common experience is used in mathematics to study congruent figures.



If the figure at the right were folded along line k, $\triangle ABC$ would coincide with $\triangle A'B'C$. Line k is the line of reflection, the image of A is A' (in symbols, $A \to A'$), and the image of B is B' ($B \to B'$). Point C is a fixed point because it is *on* the line of reflection. In other words, C is its own image ($C \to C$). Under a reflection in line k, then, the image of $\triangle ABC$ is $\triangle A'B'C$ ($\triangle ABC \to \triangle A'B'C$).



If we imagine that isosceles triangle ABC, shown at the right, is folded so that A falls on C, the line along which it folds, k, is a reflection line. Every point of the triangle has as its image a point of the triangle. Points B and D are fixed points because they are on the line of reflection.

Thus, under the line reflection in k:

1. All points of $\triangle ABC$ are reflected so that $A \to C$, $C \to A$, $E \to F$, $F \to E$, $B \to B$, $D \to D$, and so on.
2. The sides of $\triangle ABC$ are reflected; that is, $\overline{AB} \to \overline{CB}$, a statement verifying that the legs of an isosceles triangle are congruent. Also, $\overline{AC} \to \overline{CA}$, showing that the base is its own image.
3. The angles of $\triangle ABC$ are reflected; that is, $\angle BAD \to \angle BCD$, a statement verifying that the base angles of an isosceles triangle are congruent. Also, $\angle ABC \to \angle CBA$, showing that the vertex angle is its own image.

Looking at isosceles triangle ABC and reflection line k, we can note some properties of a line reflection:

1. Distance is preserved (unchanged).

$$\overline{AB} \to \overline{CB} \quad \text{and} \quad AB = CB$$

$$\overline{AD} \to \overline{CD} \quad \text{and} \quad AD = CD$$

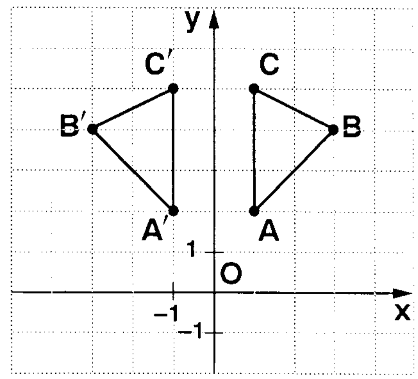
2. Angle measure is preserved.

$$\angle BAD \rightarrow \angle BCD \quad \text{and} \quad m\angle BAD = m\angle BCD$$

$$\angle BDA \rightarrow \angle BDC \quad \text{and} \quad m\angle BDA = m\angle BDC$$

3. The line of reflection is the perpendicular bisector of every segment joining a point to its image.
4. A figure is always congruent to its image.

Reflection in the y-axis In the figure, $\triangle ABC$ is reflected in the y-axis. Its image under the reflection is $\triangle A'B'C'$. From the figure, we see that:



$$A(1, 2) \rightarrow A'(-1, 2)$$

$$B(3, 4) \rightarrow B'(-3, 4)$$

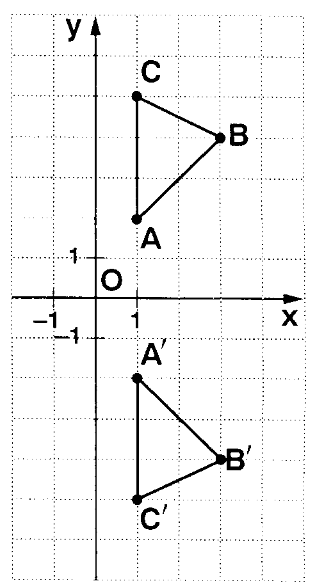
$$C(1, 5) \rightarrow C'(-1, 5)$$

For each point and its image under a reflection in the y-axis, the y-coordinate of the image is the same as the y-coordinate of the point; the x-coordinate of the image is the opposite of the x-coordinate of the point.

From these examples, we form a general rule:

- **Under a reflection in the y-axis, the image of $P(x, y)$ is $P'(-x, y)$.**

Reflection in the x-axis In the figure, $\triangle ABC$ is reflected in the x-axis. Its image under the reflection is $\triangle A'B'C'$. From the figure, we see that:



$$A(1, 2) \rightarrow A'(1, -2)$$

$$B(3, 4) \rightarrow B'(3, -4)$$

$$C(1, 5) \rightarrow C'(1, -5)$$

For each point and its image under a reflection in the x-axis, the x-coordinate of the image is the same as the x-coordinate of the point; the y-coordinate of the image is the opposite of the y-coordinate of the point.

From these examples, we form a general rule:

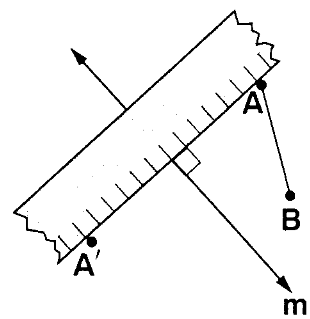
- **Under a reflection in the x-axis, the image of $P(x, y)$ is $P'(x, -y)$.**

MODEL PROBLEMS

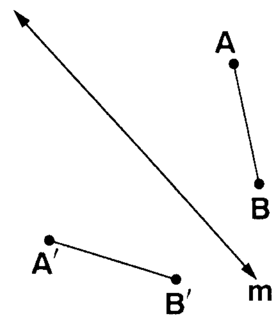
1. *a.* On your paper, draw a segment and label the endpoints A and B.
b. Draw any line m.
c. Sketch the image of $\overline{AB}$ under a reflection in m.

Solutions:

1. Draw $\overline{AB}$ and line m.
2. Hold a ruler perpendicular to line m and touching point A. Measure the distance from A to line m. Find a point along the ruler that is the same distance from m as A but that is on the opposite side of m. Label this point A'.



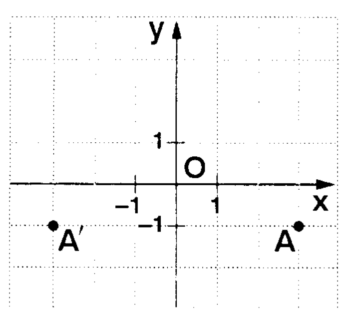
3. Repeat step 2 for point B to locate B'.
4. Draw $\overline{A'B'}$, the image of $\overline{AB}$.



2. On graph paper:
a. Plot $A(3, -1)$.
b. Plot A', the image of A under a reflection in the y-axis, and write its coordinates.
c. Plot A'', the image of A under a reflection in the x-axis, and write its coordinates.

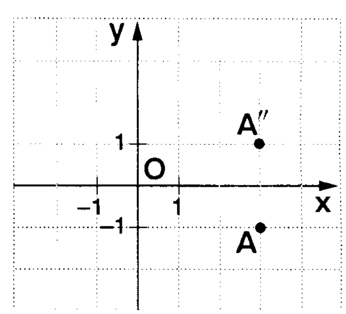
Solution:

a, b.



$A \to A'$
$(3, -1) \to (-3, -1)$

c.

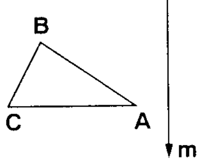


$A \to A''$
$(3, -1) \to (3, 1)$

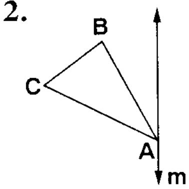
EXERCISES

In 1–4, in each case: *a.* Copy the figure and line m on your paper. *b.* Using a ruler, sketch the image of the given figure under a reflection in m.

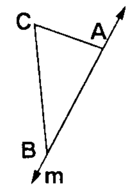
1.



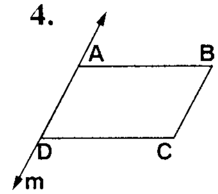
2.



3.



4.



5. *a.* Draw square $ABCD$.
b. Draw m, a line of reflection for which the image of A is B.
c. Draw n, a line of reflection for which the image of A is C.
d. Draw p, a line of reflection for which the image of A is D.

Use graph paper for exercises 6–15.

In 6–10: *a.* Graph each point and its image under a reflection in the x-axis. *b.* Write the coordinates of the image point.

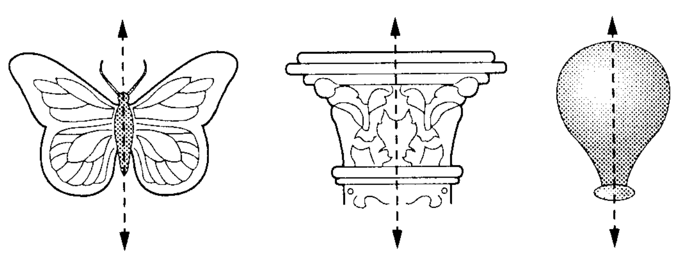
6. (2, 5) **7.** (1, 3) **8.** (−2, 3) **9.** (2, −4) **10.** (0, 2)

In 11–15: *a.* Graph each point and its image under a reflection in the y-axis. *b.* Write the coordinates of the image point.

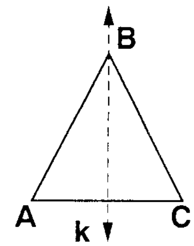
11. (3, 5) **12.** (1, 4) **13.** (2, −3) **14.** (−2, 3) **15.** (−1, 0)

Line Symmetry

In nature, in art, and in industry, many forms have a pleasing, attractive appearance because of a balanced arrangement of their parts. We say that such forms have symmetry.

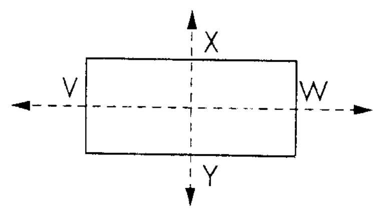


In each of the figures above, there is a line on which the figure could be folded so that the parts of the figure on opposite sides of the line would coincide. If we think of that line as a line of reflection, each point of the figure has as its image a point of the figure. This line of reflection is a line of symmetry, or ***axis of symmetry,*** and the figure has ***line symmetry.***

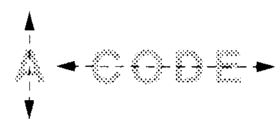


An isosceles triangle has line symmetry. In the diagram on the right, the line of reflection, k, is an axis of symmetry and isosceles triangle ABC is symmetric with respect to the line through its vertex that is perpendicular to its base.

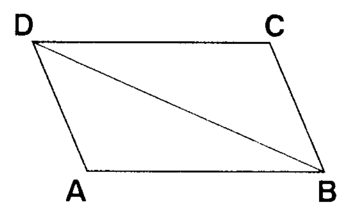
It is possible for a figure to have more than one axis of symmetry. In the rectangle at the right, line XY is an axis of symmetry and line VW is a second axis of symmetry.



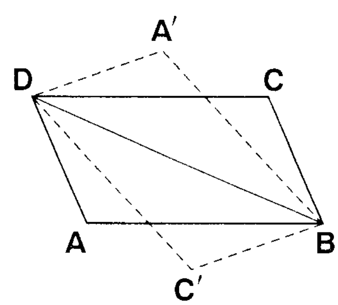
Lines of symmetry may be found for some letters and for some words, as shown at the right.



Not every figure, however, has line symmetry. If $\square ABCD$ at the right is reflected in diagonal $\overline{BD}$, the image of A is A' and the image of C is C'. Points A' and C' however, are not points of the original parallelogram. The image of $\square ABCD$ under a reflection in $\overline{BD}$ is $\square A'BC'D$. Therefore, $\square ABCD$ is not symmetric with respect to $\overline{BD}$.



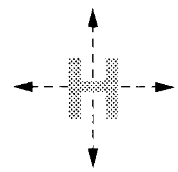
We have used diagonal $\overline{BD}$ as a line of reflection, but note that it is not a line of symmetry. In other words, there is no line along which the parallelogram can be folded so that points of the parallelogram on one side of the line will coincide with points of the parallelogram on the other.



MODEL PROBLEMS

1. How many lines of symmetry does the letter **H** have?

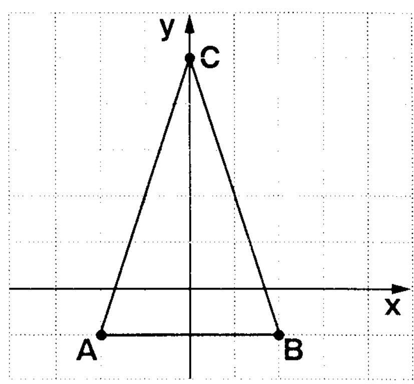
Solution: The horizontal line through the crossbar is a line of symmetry. The vertical line midway between the vertical segments is also a line of symmetry.



Answer: The letter **H** has two lines of symmetry.

2. Draw $\triangle ABC$ whose vertices are $A(-2, -1)$, $B(2, -1)$, $C(0, 5)$. What line is a line of symmetry for the triangle?

Solution:



Answer: The y-axis is a line of symmetry for $\triangle ABC$.

EXERCISES

1. Using the printed capital letters of the alphabet, write on your paper all letters that have line symmetry. Indicate the lines of symmetry.
2. Copy each of the following "words," and draw a line of symmetry, or indicate that the word does not have line symmetry by writing "None."

a. MOM	*b.* DAD	*c.* SIS	*d.* OTTO
e. BOOK	*f.* RADAR	*g.* un	*h.* NOON
i. HIKE	*j.* SWIMS	*k.* OHHO	*l.* CHOKED

In 3–14, for each geometric figure named: *a.* Sketch the figure. *b.* Tell the number of lines of symmetry, if any, that the figure has, and sketch them on your drawing.

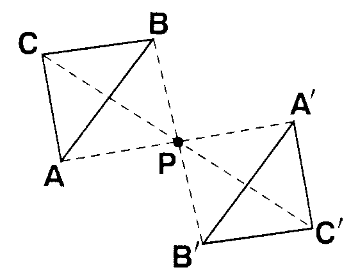
3. rectangle
4. equilateral triangle
5. parallelogram
6. isosceles triangle
7. rhombus
8. regular hexagon
9. trapezoid
10. scalene triangle
11. circle
12. regular octagon
13. square
14. regular pentagon

15. Draw rectangle $PQRS$, whose vertices are $P(-5, -2)$, $Q(5, -2)$, $R(5, 2)$, and $S(-5, 2)$. What two lines are lines of symmetry for the rectangle?

16. Draw rectangle $ABCD$, whose vertices are $A(2, 0)$, $B(2, 5)$, $C(4, 5)$, and $D(4, 0)$. Draw the axes of symmetry for the rectangle.

14. Point Reflections and Point Symmetry

Another kind of reflection involves a point. In the figure at the right, $\triangle A'B'C'$ is the image of $\triangle ABC$ under a reflection in point P. If a line segment is drawn connecting any point to its image, then the point of reflection is the midpoint of that segment. In the figure:

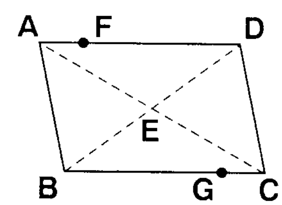


Point A' is on $\overleftrightarrow{AP}$, $AP = PA'$, and P is the midpoint of $\overline{AA'}$.

Point B' is on $\overleftrightarrow{BP}$, $BP = PB'$, and P is the midpoint of $\overline{BB'}$.

Point C' is on $\overleftrightarrow{CP}$, $CP = PC'$, and P is the midpoint of $\overline{CC'}$.

In parallelogram $ABCD$, shown at the right, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. Point E is the midpoint of $\overline{AC}$ and of $\overline{BD}$. Therefore, under a reflection in point E, $A \to C$ and $C \to A$, $B \to D$ and $D \to B$. Similarly, $F \to G$ and $G \to F$, and every point of the parallelogram has its image on the parallelogram.



Properties of Point Reflections

Looking at parallelogram $ABCD$ and point of reflection E, we can note some properties of ***point reflection***:

1. Distance is preserved.

$$\overline{AB} \to \overline{CD} \quad \text{and} \quad AB = CD$$

$$\overline{AD} \to \overline{CB} \quad \text{and} \quad AD = CB$$

2. Angle measure is preserved.

$$\angle BAD \rightarrow \angle DCB \quad \text{and} \quad m\angle BAD \rightarrow m\angle DCB$$
$$\angle ABC \rightarrow \angle CDA \quad \text{and} \quad m\angle ABC \rightarrow m\angle CDA$$

3. The point of reflection is the midpoint of every segment formed by joining a point to its image.

$$AE = EC \text{ and } BE = ED$$

4. A figure is always congruent to its image.

Reflection in the Origin

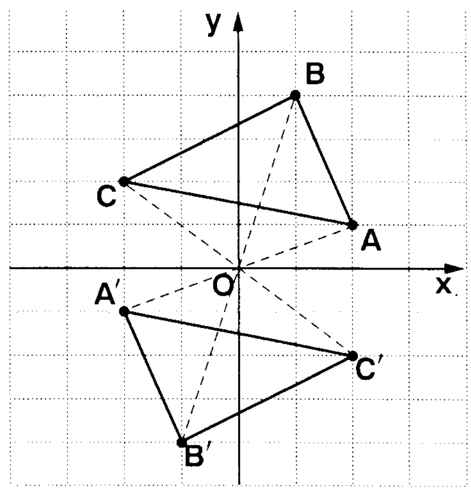
The origin, $O(0, 0)$, is the most common point of reflection in the coordinate plane.

$A(2, 1) \rightarrow A'(-2, -1)$

$B(1, 4) \rightarrow B'(-1, -4)$

$C(-2, 2) \rightarrow C'(2, -2)$

The image of $\triangle ABC$ under a reflection in the origin is $\triangle A'B'C'$.



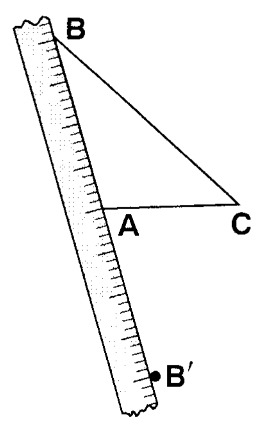
MODEL PROBLEMS

1. On your paper, draw any triangle ABC. Sketch the image of $\triangle ABC$ under a reflection in point A.

Solution:

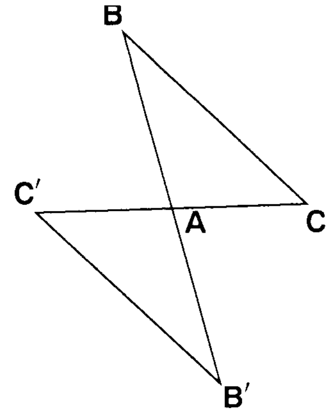
1. Draw a triangle and label it $\triangle ABC$.

2. Hold a ruler on $\overline{AB}$ and measure the distance from B to A. Since A is the point of reflection, the image of B is on the line $\overleftrightarrow{AB}$. Locate the point along the ruler that is the same distance from A as B but that is on the opposite side of A. Label it point B'.



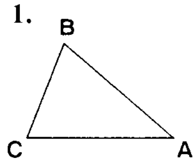
3. Repeat step 2 for C by placing the ruler on $\overline{AC}$ to locate point C'.

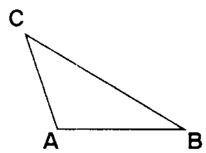
4. Draw $\triangle AB'C'$, the image of $\triangle ABC$.

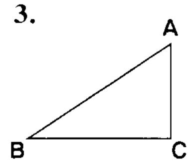


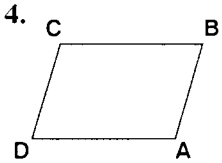
EXERCISES

In 1–4, on your paper, copy each figure. Using a ruler, sketch the image of the figure under a reflection in point A.

1.


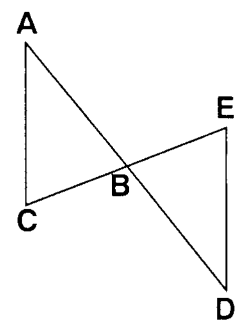
2.


3.


4.


5. In the figure, $\triangle ABC \cong \triangle DBE$. Find the image of each of the following under a reflection in B:

a. A	*b.* B	*c.* C
d. D	*e.* E	*f.* $\overline{AC}$
g. $\overline{AB}$	*h.* $\overline{DE}$	

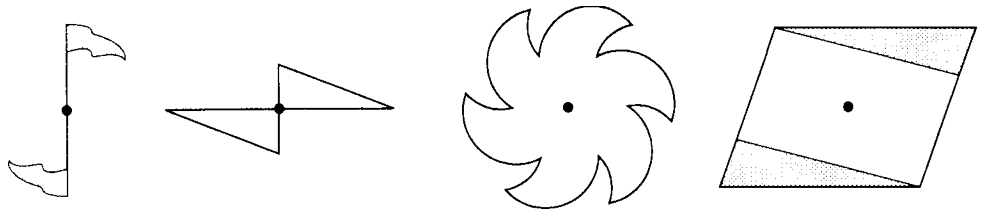


In 6–9, in each case: *a.* Graph the point and its reflection in the origin. *b.* State the coordinates of the image point.

6. $A(4, 3)$ **7.** $B(-3, 2)$ **8.** $C(-2, -1)$ **9.** $D(5, -4)$

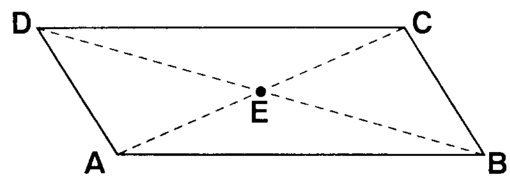
Point Symmetry

In each of the figures shown below, the design is built around a central point. For every point in the figure, there is another point at the same distance from the center, so that the center is the midpoint of the segment joining the

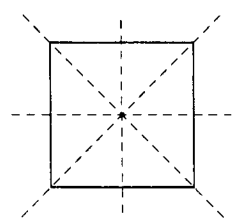


pair of points. Under a point reflection through the center, each point has as its image, another point of the figure. The figure has ***point symmetry.***

Parallelogram *ABCD* at the right has point symmetry under a reflection in point *E*, the intersection of its diagonals.



It is possible for a figure to have both line symmetry and point symmetry at the same time. In the square at the right, there are four lines of symmetry. Note that the point of symmetry lies at the intersection of these four lines.

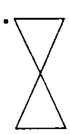
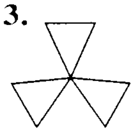
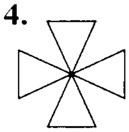
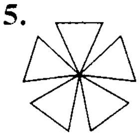
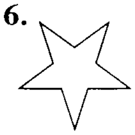
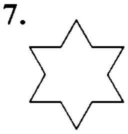


Points of symmetry may be found for some letters and for some words, as shown at the right.

EXERCISES

1. *a.* Using the printed capital letters of the alphabet, write on your paper all of the letters that have point symmetry.
b. Show the point of symmetry for each letter.

In 2–7, tell whether each figure has point symmetry.

2.  **3.**  **4.**  **5.**  **6.**  **7.** 

8. Which of the following has point symmetry?

a. WOW	*b.* O	*c.* SIS	*d.* OTTO
e. pop	*f.* pod	*g.* un	*h.* NOON
i. HOHO	*j.* OHHO	*k.* SWIMS	*l.* SOS

9. Draw a figure that has point symmetry but does not have line symmetry.

10. Draw a figure that has both point symmetry and line symmetry.

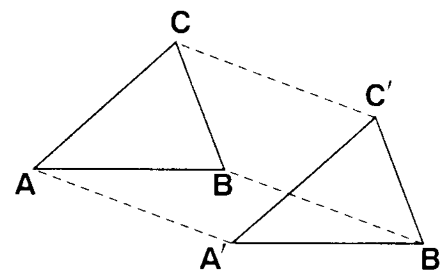
11. Which polygons have point symmetry?
   a. All polygons
   b. All regular polygons
   c. Only squares
   d. Regular polygons with an even number of sides

15. Translations

It is often useful or necessary to move objects from one place to another. If we move a desk from one place in the room to another, each leg moves the same distance in the same direction.

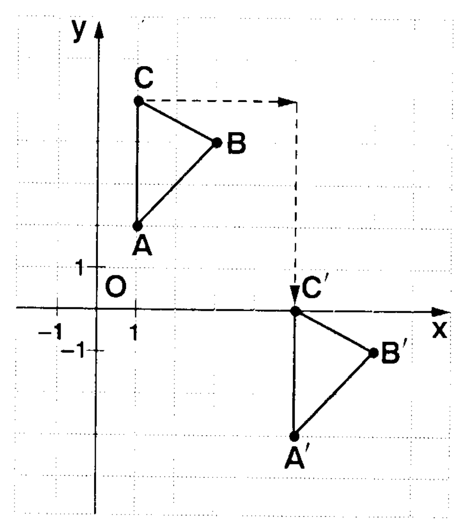
A ***translation*** moves every point in the plane the same distance in the same direction.

If $\triangle A'B'C'$ is the image of $\triangle ABC$ under a translation, $AA' = BB' = CC'$ and $\overline{AA'} \parallel \overline{BB'} \parallel \overline{CC'}$. The size and shape of the figure are unchanged, so that $\triangle ABC \cong \triangle A'B'C'$. Thus, as with reflections, a figure is congruent to its image under a translation.



In the figure at the right, $\triangle ABC$ is translated by moving every point 4 units to the right and 5 units down. From the figure, we see that:

$$A(1, 2) \to A'(5, -3)$$
$$B(3, 4) \to B'(7, -1)$$
$$C(1, 5) \to C'(5, 0)$$

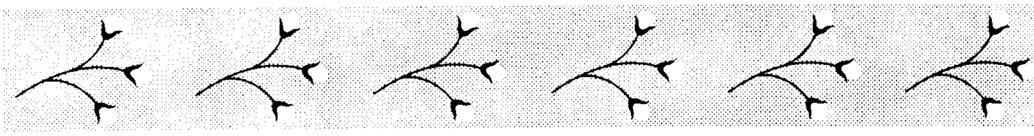


For each point and its image under a translation that moves every point 4 units to the right (+4) and 5 units down (−5), the x-coordinate of the image is 4 more than the x-coordinate of the point ($x \to x + 4$); the y-coordinate of the image is 5 less than the y-coordinate of the point ($y \to y - 5$).

From this example, we form a general rule:

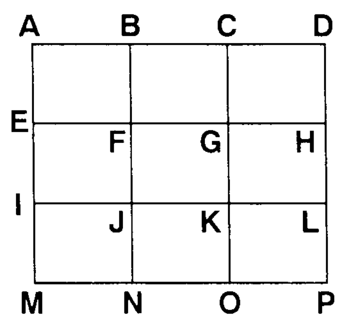
- **Under a translation of a units in the horizontal direction and b units in the vertical direction, the image of $P(x, y)$ is $P'(x + a, y + b)$.**

Patterns used for decorative purposes such as wallpaper or borders on clothing often appear to have ***translational symmetry.*** True translational symmetry would be possible, however, only if the pattern could repeat without end.



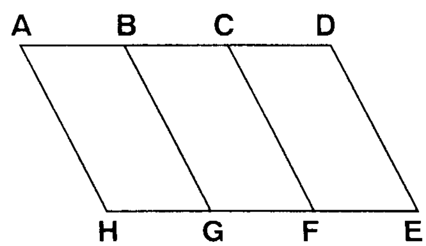
EXERCISES

For 1 and 2, the diagram consists of nine congruent rectangles.



Ex. 1–2

1. Under a translation, the image of A is G. Find the image of each of the given points under the same translation.
   a. J *b.* B *c.* I *d.* F *e.* E
2. Under a translation, the image of K is J. Find the image of each of the given points under the same translation.
   a. J *b.* B *c.* O *d.* L *e.* G



3. In the diagram, $ADEH$ is a parallelogram. Points B, C, G, and F divide $\overline{AD}$ and $\overline{EH}$ into congruent segments. Under a translation, the image of A is G. Under the same translation, tell whether or not the image of each given point is a point of the diagram.
   a. G *b.* B *c.* C *d.* F
4. *a.* On graph paper, draw and label $\triangle ABC$, whose vertices have the coordinates $A(1, 2)$, $B(6, 3)$, and $C(4, 6)$.
   b. Under the translation $P(x, y) \rightarrow P'(x + 5, y - 3)$, every point moves 5 units to the right and 3 units down. For example, under this translation, the image of $A(1, 2)$ is $A'(1 + 5, 2 - 3)$ or $A'(6, -1)$. If, under this translation, the image of B is B' and the image of C is C', find the coordinates of B' and C'.
   c. On the same graph drawn in part *a*, draw and label $\triangle A'B'C'$.

5. *a.* On graph paper, draw and label $\triangle ABC$ if the coordinates of A are $(-2, -2)$, the coordinates of B are $(2, 0)$, and the coordinates of C are $(3, -3)$.

b. On the same graph, draw and label $\triangle A'B'C'$, the image of $\triangle ABC$ under a translation whose rule is $(x, y) \to (x - 4, y + 7)$.

c. Give the coordinates of the vertices of $\triangle A'B'C'$.

6. Which of the following is the rule of the translation in which every point moves 6 units to the right on a graph?

a. $(x, y) \to (x, y + 6)$ *b.* $(x, y) \to (x + 6, y)$

c. $(x, y) \to (x + 6, y + 6)$ *d.* $(x, y) \to (x - 6, y)$

7. In a translation, every point moves 4 units down. Write a rule for this translation by completing the sentence $(x, y) \to \ldots$.

8. The coordinates of $\triangle ABC$ are $A(2, 1)$, $B(4, 1)$, and $C(5, 5)$.

a. On graph paper, draw and label $\triangle ABC$.

b. Write a rule for the translation in which the image of A is $C(5, 5)$.

c. Use the rule from part *b* to find the coordinates of B', the image of B, and C', the image of C, under this translation.

d. On the graph drawn in part *a,* draw and label $\triangle CB'C'$, the image of $\triangle ABC$.

9. The coordinates of $\triangle ABC$ are $A(2, 1)$, $B(4, 1)$, and $C(5, 5)$.

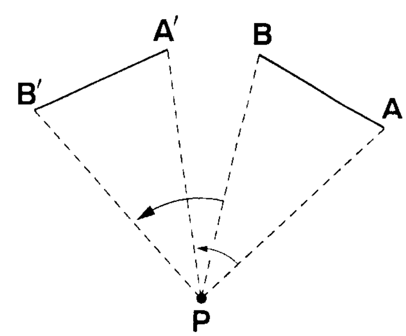
a. On graph paper, draw and label $\triangle ABC$.

b. Under a translation, the image of C, C', is at $B(4, 1)$. Find the coordinates of A', the image of A, and of B', the image of B, under this same translation.

c. On the graph drawn in part *a,* draw and label $\triangle A'B'C'$, the image of $\triangle ABC$.

d. How many points, if any, are fixed points under this translation?

16. Rotations

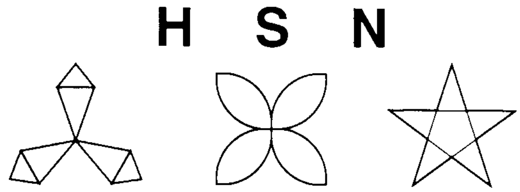


Think of what happens to all of the points of the steering wheel of a car as the wheel is turned. Except for the fixed point in the center, every point moves through a part of a circle, or arc, so that the position of each point is changed by a ***rotation*** of the same number of degrees.

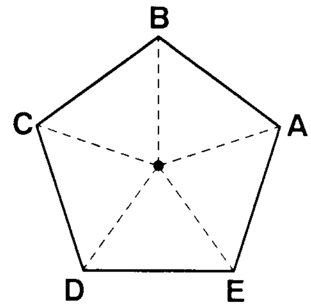
In the figure, if A is rotated to A', then B is rotated the same number of degrees to B', and $m\angle APA' = m\angle BPB'$. Since P is the center of rotation, $PA = PA'$ and $PB = PB'$.

In general, a rotation preserves distance and angle measure. Under a rotation, a figure is congruent to its image. Unless otherwise stated, a rotation is in the counterclockwise direction.

Many letters, as well as designs in the shapes of wheels, stars, and polygons, have ***rotational symmetry.*** Each figure shown at the right has rotational symmetry.



Any regular polygon has rotational symmetry. When regular pentagon $ABCDE$ is rotated $\frac{360°}{5}$, or 72°, about its center, the image of every point of the figure is a point of the figure. Under this rotation, $A \to B$, $B \to C$, $C \to D$, $D \to E$, and $E \to A$.

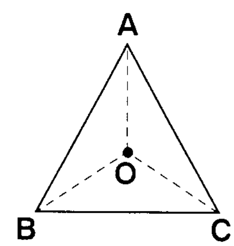


The figure would also have rotational symmetry if rotated through a multiple of 72° (144°, 216°, or 288°). If it were rotated through 360°, every point would be its own image. Since this is true for every figure, we do not usually consider a 360° rotation as rotational symmetry.

MODEL PROBLEM

Point O is at the center of equilateral triangle ABC so that $OA = OB = OC$. Find the image of each of the following under a rotation of 120° about O:

a. A *b.* B *c.* C *d.* $\overline{AB}$ *e.* $\angle CAB$



Answers:

a. B *b.* C *c.* A *d.* $\overline{BC}$ *e.* $\angle ABC$

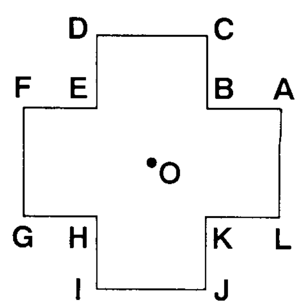
EXERCISES

1. What is the image of each of the given points under a rotation of 90° in the counterclockwise direction about O?

a. A *b.* B *c.* C *d.* G *e.* H *f.* J *g.* K

2. What is the image of each of the given points under a rotation of 90° in the clockwise direction about O?

a. A *b.* B *c.* C *d.* G *e.* H *f.* J *g.* K



Ex. 1–2

In 3–8, for each geometric figure named:

a. Sketch the figure.
b. If the figure has rotational symmetry, mark the center of rotation with a dot.
c. If the figure has rotational symmetry, give the measure of the smallest angle for which the symmetry exists.

3. Rectangle **4.** Parallelogram **5.** Rhombus
6. Trapezoid **7.** Regular hexagon **8.** Regular pentagon

17. Compositions of Transformations

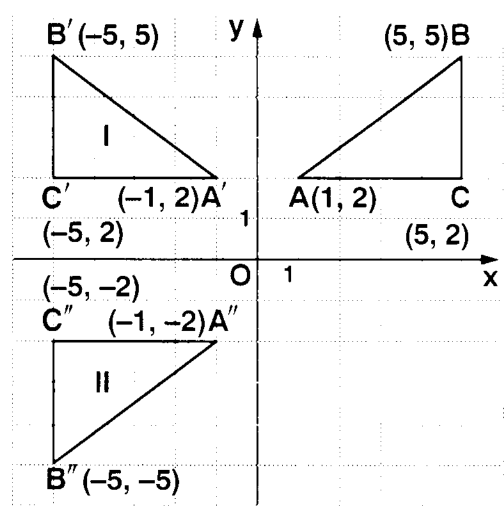
The combination of two transformations is called a ***composition of transformations*** when the first transformation produces an image, and the second transformation is performed on that image.

We start again with $\triangle ABC$, whose vertices are $A(1, 2)$, $B(5, 5)$, and $C(5, 2)$. For the composition of transformations, we will consider two line reflections. First, by reflecting $\triangle ABC$ in the y-axis, we form $\triangle A'B'C'$, or $\triangle$I. Then, by reflecting $\triangle A'B'C'$ in the x-axis, we form $\triangle A''B''C''$, or $\triangle$II. We observe how the vertices are related:

$A(1, 2) \rightarrow A'(-1, 2) \rightarrow A''(-1, -2)$

$B(5, 5) \rightarrow B'(-5, 5) \rightarrow B''(-5, -5)$

$C(5, 2) \rightarrow C'(-5, 2) \rightarrow C''(-5, -2)$



Now let us compare the original triangle, $\triangle ABC$, and the final image, $\triangle A''B''C''$, with the image of $\triangle ABC$ under a reflection in the origin. We observe:

- **The composition of a line reflection in the y-axis, followed by a line reflection in the x-axis, is equivalent to a single transformation, namely, a reflection through point O, the origin.**

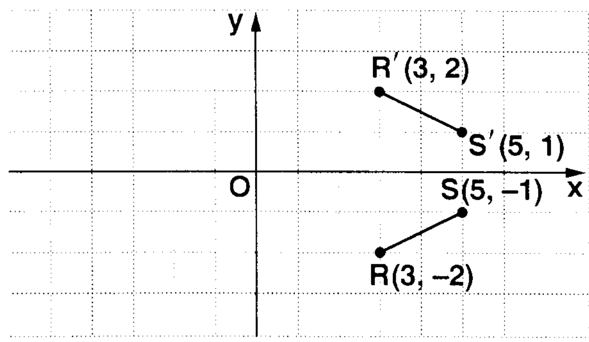
We reflected the triangle first in the y-axis and then in the x-axis. If we had reflected the triangle first in the x-axis and then in the y-axis, would this composition also be equivalent to a reflection through point O, the origin? The answer is yes. However, not all compositions will act in the same way. In general, compositions of transformations are *not* commutative.

MODEL PROBLEMS

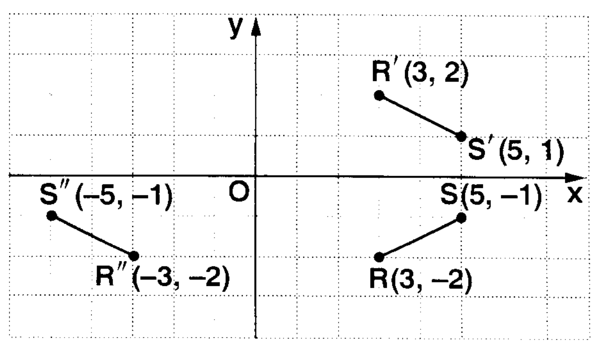
1. The coordinates of the endpoints of $\overline{RS}$ are $R(3, -2)$ and $S(5, -1)$.
   a. Graph the line segment $\overline{RS}$.
   b. Graph on the same set of axes the image $\overline{R'S'}$, a reflection of $\overline{RS}$ in the x-axis.
   c. Graph on the same set of axes $\overline{R''S''}$, a reflection of $\overline{R'S'}$ in the origin.
   d. Name the single transformation by which $\overline{RS} \rightarrow \overline{R''S''}$.

Solutions:

a, b. $R(3, -2) \rightarrow R'(3, 2)$
$S(5, -1) \rightarrow S'(5, 1)$



c. $R'(3, 2) \rightarrow R''(-3, -2)$
$S'(5, 1) \rightarrow S''(-5, -1)$

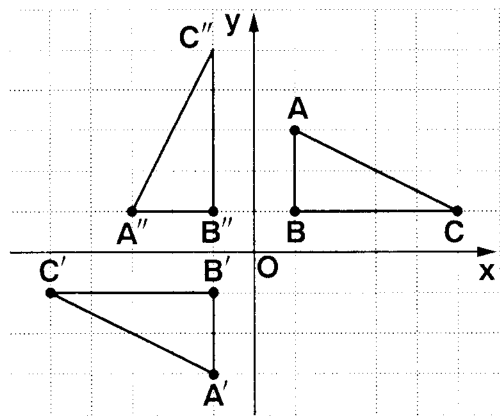


d. $\overline{RS} \rightarrow \overline{R''S''}$ is the same as a reflection of $\overline{RS}$ in the y-axis.
$R(3, -2) \rightarrow R''(-3, -2)$
$S(5, -1) \rightarrow S''(-5, -1)$

2. The coordinates of $\triangle ABC$ are $A(1, 3)$, $B(1, 1)$, and $C(5, 1)$. Is the image $\triangle A''B''C''$ the same if it is reflected in the origin, then rotated 90° clockwise about the origin as the image is if it is rotated 90° clockwise about the origin, then reflected 90° in the origin? Explain your answer.

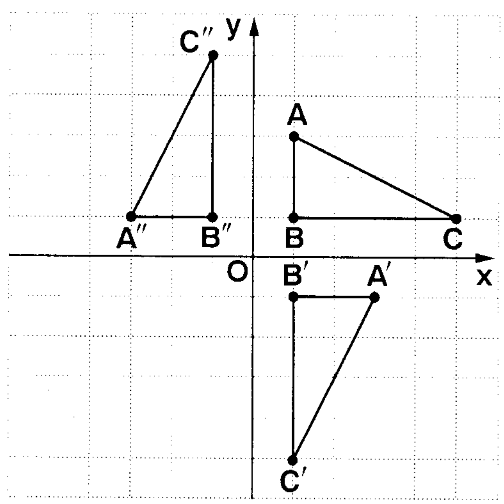
Solution: Under a reflection in the origin, then rotated 90° clockwise about the origin:

$A(1, 3) \to A'(-1, -3)$	$A'(-1, -3) \to A''(-3, 1)$
$B(1, 1) \to B'(-1, -1)$	$B'(-1, -1) \to B''(-1, 1)$
$C(5, 1) \to C'(-5, -1)$	$C'(-5, -1) \to C''(-1, 5)$



Rotated 90° clockwise about the origin, then under a reflection in the origin:

$A(1, 3) \to A'(3, -1)$	$A'(3, -1) \to A''(-3, 1)$
$B(1, 1) \to B'(1, -1)$	$B'(1, -1) \to B''(-1, 1)$
$C(5, 1) \to C'(1, -5)$	$C'(1, -5) \to C''(-1, 5)$



In each case $\triangle A''B''C''$ is the same.

EXERCISES

1. On graph paper draw $\triangle ABC$ whose vertices are $A(-5, 2)$, $B(-3, 6)$, and $C(0, 0)$.

a. Using the same set of axes, graph $\triangle A'B'C'$, the image of $\triangle ABC$ under a reflection in the origin.

b. Using the same set of axes, graph $\triangle A''B''C''$, a reflection of $\triangle A'B'C'$ in the y-axis.

c. Name the single transformation by which $\triangle ABC \rightarrow \triangle A''B''C''$.

In 2 and 3, the vertices of $\triangle LMN$ are $L(1, 3)$, $M(1, 1)$, and $N(5, 1)$.

2. *a.* Draw $\triangle LMN$ and its image, $\triangle L'M'N'$, under a point reflection through the origin.

b. Using the same graph, reflect $\triangle L'M'N'$ in the x-axis to form its image, $\triangle L''M''N''$.

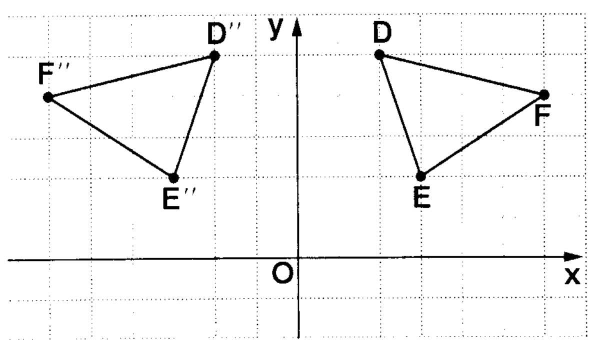
c. What single transformation is equivalent to the composition of a point reflection through the origin followed by a reflection in the x-axis?

3. *a.* Draw $\triangle LMN$ and its image $\triangle L'M'N'$ under a reflection in the x-axis.

b. Using the same graph, reflect $\triangle L'M'N'$ in the y-axis to form its image, $\triangle L''M''N''$.

c. What single transformation is equivalent to the composition of a reflection in the x-axis followed by a reflection in the y-axis?

4. Given $\triangle DEF$ and $\triangle D''E''F''$ as shown on the grid.



a. Describe one pair of transformations that will result in $\triangle DEF \rightarrow \triangle D''E''F''$.

b. Name a single transformation that will give the same result as the pair described in *a.*

5. On graph paper draw parallelogram $RSTU$ with vertices $R(-8, -4)$, $S(-6, -2)$, $T(-3, -2)$, and $U(-5, -4)$.

a. Find the coordinates of the point of symmetry for parallelogram $RSTU$.

b. Find the images of $\overline{RS}$ and $\overline{RU}$ under this point reflection.

6. The vertices of parallelogram $ABCD$ are $A(1, 1)$, $B(3, 5)$, $C(9, 5)$, and $D(7, 1)$.

a. Find the coordinates of the point of symmetry for parallelogram $ABCD$.

b. Find the images of $\overline{AB}$ and $\overline{AD}$ under this point reflection.

7. The coordinates of the endpoints of $\overline{YZ}$ are $Y(3, 4)$ and $Z(1, 1)$. Is the image $\overline{Y''Z''}$ the same if it is reflected in the y-axis, then rotated 90° counterclockwise about the origin as the image is if it is rotated 90° counterclockwise about the origin then reflected in the y-axis? Explain your answer.

8. The coordinates of $\triangle JKL$ are $J(2, 4)$, $K(2, 2)$, and $L(6, 2)$. Is the image $\triangle J''K''L''$ the same if it is reflected in the origin, then rotated 180° counterclockwise about the origin as the image is if it is rotated 180° counterclockwise about the origin then reflected in the origin? Explain your answer.