CHAPTER V

Circles

1. Definitions and Fundamental Relations

Previously, on page 24, we learned the definition of a *circle*. We also learned the definitions of *radius*, *chord*, and *diameter*, which are line segments associated with the circle. In circle O (Fig. 5–1), $\overline{OC}$ is a radius, $\overline{AB}$ is a diameter, and $\overline{DE}$ is a chord. Now we will define several additional terms.

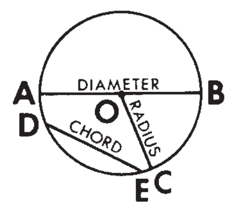


Fig. 5–1

Arcs and Central Angles

Definition. **A *central angle* of a circle is an angle whose vertex is the center of the circle.**

In circle O (Fig. 5–2), $\angle AOB$ is a central angle. We say that central $\angle AOB$ "intercepts," or "has," the arc $\widehat{AB}$. We also say that arc $\widehat{AB}$ "subtends," or "has," the central $\angle AOB$.

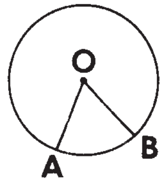


Fig. 5–2

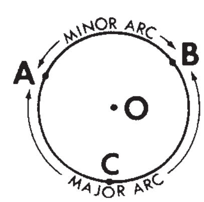


Fig. 5–3

Definition. **A *minor arc* of a circle is the union of two points of the circle that are not the ends of a diameter and the set of points of the circle which lie in the interior of the central angle whose sides contain the two points.**

In circle O (Fig. 5–3), "arc AB," or $\widehat{AB}$, is a minor arc.

***Definition.* A *major arc* of a circle is the union of two points of a circle that are not the ends of a diameter and the set of points of the circle which lie in the exterior of the central angle whose sides contain the two points.**

In circle O (Fig. 5–3), "arc ACB," or $\widehat{ACB}$, is a major arc.

NOTE. When we refer to an arc, we will mean the minor arc, unless we state otherwise.

We say that a chord "subtends" the arcs that it cuts off from a circle. Unless we state otherwise, the arc subtended by a chord will always mean the minor arc subtended by the chord.

The Measure of an Arc

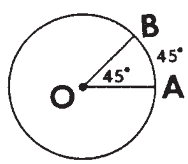


Fig. 5–4

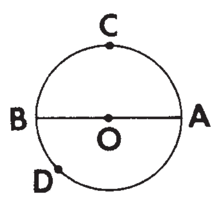


Fig. 5–5

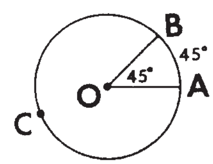


Fig. 5–6

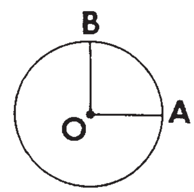


Fig. 5–7

***Definition.* The *measure of a minor arc* is the measure of the central angle that intercepts the arc.**

In circle O (Fig. 5–4), the measure of arc $\widehat{AB}$, symbolized by $m\widehat{AB}$, is the measure of the central angle AOB, or $m\angle AOB$. If $m\angle AOB = 45$, then $m\widehat{AB} = 45$.

***Definition.* The *measure of a semicircle* is 180.**

In circle O (Fig. 5–5), the measure of semicircle $\widehat{ACB}$, symbolized by $m\widehat{ACB}$, is 180. Also, $m\widehat{BDA}$ is 180. Hence, the entire circle will have a measure of 360.

***Definition.* The *measure of a major arc* is 360 minus the measure of the minor arc which has the same endpoints as the major arc.**

In circle O (Fig. 5–6), the measure of major arc BCA, symbolized by $m\widehat{BCA}$, is 360 minus the measure of $\widehat{BA}$. If $m\angle AOB = 45$, then $m\widehat{BA} = 45$. Hence, $m\widehat{BCA} = 360 - 45 = 315$.

***Definition.* A *quadrant* is an arc whose measure is 90.**

In circle O (Fig. 5–7), $\widehat{AB}$ is a quadrant if $m\widehat{AB} = 90$.

Angle Degrees and Arc Degrees

We have learned that $m\angle AOB = 5$ (Fig. 5–8) indicates that $\angle AOB$ is an angle of 5 degrees where the unit of measure for the size of an angle is the degree. Likewise, we will write $m\widehat{AB} = 5$ to indicate that $\widehat{AB}$ is an arc of 5 degrees where the unit of measure for the size of an arc is the degree. However, to distinguish the unit of measure for the angle from that of the arc, we will call the former an *angle degree*, whereas we will call the latter an *arc degree*.

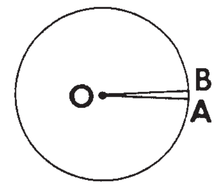


Fig. 5–8

Linear Measure of an Arc

In Fig. 5–9, we see that $m\angle AOB = 45$; therefore, $\widehat{AB}$ contains 45 arc degrees. We also see that $m\angle COD = 45$; therefore, $\widehat{CD}$ contains 45 arc degrees. Although both arcs contain 45 arc degrees, they are not equal in length; that is, they do not have the same *linear measure* because the length of $\widehat{AB}$ is $\frac{45}{360}$, or $\frac{1}{8}$, of the length of the smaller circle, whereas the length of $\widehat{CD}$ is $\frac{45}{360}$, or $\frac{1}{8}$, of the length of the larger circle.

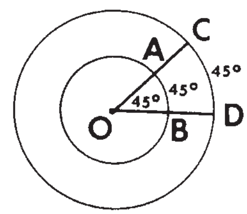


Fig. 5–9

In a later chapter, we will learn how to compute the length of a circle and the length of an arc. For the present, we can find the approximate length of an arc by fitting a string tightly around the arc and measuring the length of the piece that fits around the arc. We assume that with every arc there is associated a number which expresses the length of the arc in terms of a unit such as inches or feet. We will symbolize "the length of arc AB" by $\widehat{AB}$. Previously, we indicated that $\widehat{AB}$ means arc $\widehat{AB}$ itself. In the future, the context in which the symbol $\widehat{AB}$ is used will determine whether it represents arc $\widehat{AB}$ itself, or the length of arc $\widehat{AB}$.

Caution: Do not confuse the number of degrees contained in an arc with the length of the arc.

The Sum of Two Arcs

Consider two arcs in a circle the sum of whose measures is less than 360. By the sum of these two arcs, we will mean an arc of the circle whose length is equal to the sum of the lengths of the two arcs.

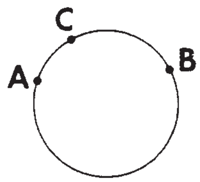


Fig. 5-10

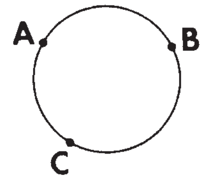


Fig. 5-11

In Fig. 5-10 and Fig. 5-11, if the intersection of $\overset{\frown}{AC}$ and $\overset{\frown}{CB}$ is the unique point C, then $\overset{\frown}{AC}$ and $\overset{\frown}{CB}$ have as their sum $\overset{\frown}{ACB}$. Thus, length of $\overset{\frown}{AC}$ + length of $\overset{\frown}{CB}$ = length of $\overset{\frown}{ACB}$. We may write this statement in a more concise form as follows: $\overset{\frown}{AC} + \overset{\frown}{CB} = \overset{\frown}{ACB}$.

Congruent Circles, Equal Circles, Congruent Arcs, and Equal Arcs

***Definition. Congruent circles* are circles whose radii are congruent.**

In Fig. 5-12, if radius $\overline{OA} \cong$ radius $\overline{O'A'}$, then circle $O \cong$ circle O'.

***Definition. Equal circles* are circles whose radii are equal in length.**

Fig. 5-12

In Fig. 5-12, if $OA = O'A'$, then circle $O =$ circle O'.

From the definitions of a circle, congruent circles, and equal circles, the following relationships, which we will use in future proofs, readily follow:

1. In a circle or in congruent circles, radii are congruent and diameters are congruent.
2. In a circle or in equal circles, radii are equal in length and diameters are equal in length.

***Definition. Congruent arcs* are arcs that have equal degree measures (arc measures) and equal lengths (linear measures).**

In circle O (Fig. 5-13), if $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ have equal degree measures, $m\overset{\frown}{AB} = m\overset{\frown}{CD}$, and $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ have equal lengths, $\overset{\frown}{AB} = \overset{\frown}{CD}$, then $\overset{\frown}{AB}$ is congruent to $\overset{\frown}{CD}$, $\overset{\frown}{AB} \cong \overset{\frown}{CD}$.

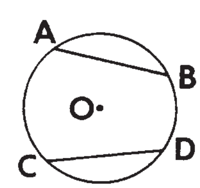


Fig. 5-13

***Postulate 36.* In the same circle or in equal circles, arcs that have equal degree measures have equal lengths.**

In circle O (Fig. 5–13), if $m\overparen{AB} = m\overparen{CD}$, then $\overparen{AB} = \overparen{CD}$.

Hence, in a circle or in equal circles, when the degree measures of two arcs are equal, or when the lengths of two arcs are equal, we will say that the two arcs are equal. Furthermore, we can also say that in the same circle or in equal circles, when two arcs are equal, the arcs are congruent. In the future, when we write that in a circle or in equal circles, $\overparen{AB} = \overparen{CD}$, the context will determine whether we mean $m\overparen{AB} = m\overparen{CD}$ or length of $\overparen{AB}$ = length of $\overparen{CD}$.

In dealing with theorems concerning arcs, we will at times talk about *equal* arcs rather than *congruent* arcs. We will do this in order to be able to make use of the addition, subtraction, multiplication and division properties of equality. Thus, if we are discussing arcs which are in the same circle or in equal circles, we may say *"equal arcs"* rather than *"congruent arcs."*

***Definition.* The *midpoint of an arc* is the point that divides the arc into two congruent arcs or two equal arcs.**

In circle O (Fig. 5–14), if $\overparen{AC} \cong \overparen{CB}$ or $\overparen{AC} = \overparen{CB}$, then C is the midpoint of $\overparen{AB}$.

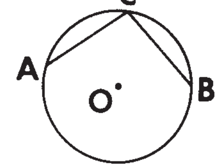


Fig. 5–14

Circle Circumscribed About a Polygon

***Definition.* A *circle circumscribed about a polygon* is a circle that passes through each vertex of the polygon.**

If circle O (Fig. 5–15) passes through every vertex of polygon $ABCD$, then the circle is circumscribed about polygon $ABCD$. We can also say that polygon $ABCD$ is "inscribed in circle O."

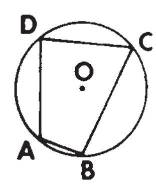
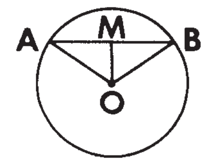


Fig. 5–15

MODEL PROBLEM

In circle O, M is the midpoint of chord $\overline{AB}$. Prove that $\overline{OM}$ bisects $\angle AOB$.



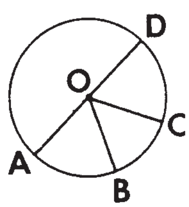
Given: Circle O.
$\overline{AM} \cong \overline{MB}$.

To prove: $\angle AOM \cong \angle BOM$.

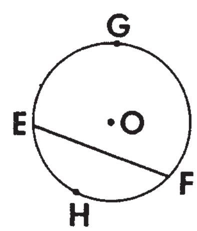
Plan: To prove that $\angle AOM \cong \angle BOM$, prove that $\triangle AOM$ and $\triangle BOM$, which contain $\angle AOM$ and $\angle BOM$, are congruent by s.s.s. $\cong$ s.s.s.

Proof: *Statements*	*Reasons*
1. $\overline{AM} \cong \overline{MB}$. (s. ≅ s.)	1. Given.
2. $\overline{OM} \cong \overline{OM}$. (s. ≅ s.)	2. Reflexive property of congruence.
3. $\overline{OA} \cong \overline{OB}$. (s. ≅ s.)	3. Radii of a circle are congruent.
4. $\triangle AOM \cong \triangle BOM$.	4. s.s.s. ≅ s.s.s.
5. $\angle AOM \cong \angle BOM$.	5. Corresponding parts of congruent triangles are congruent.

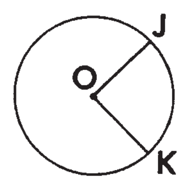
EXERCISES



Ex. 1



Ex. 2



Ex. 3

1. Name (*a*) a radius (*b*) a diameter (*c*) a semicircle.

2. Name (*a*) a chord (*b*) a minor arc (*c*) a major arc.

3. Name (*a*) a central angle (*b*) the arc intercepted by $\angle KOJ$.

4. If circle O = circle O' and the length of radius $\overline{OA}$ is 10, find the length of radius $\overline{O'A'}$.

5. In circle O, radii $\overline{OA}$ and $\overline{OB}$ are drawn. If $OA = 5x + 6$ and $OB = 2x + 24$, find the length of a radius of the circle.

6. In circle O, radii $\overline{OA}$, $\overline{OB}$, and chord $\overline{AB}$ are drawn. If $\overline{OA} = 2x + 8$, $\overline{OB} = x + 24$, and $\overline{AB} = 3x - 8$, find (*a*) the length of $\overline{OA}$, (*b*) the length of $\overline{AB}$, and (*c*) the measure of $\angle AOB$.

7. In circle O, radii $\overline{OR}$ and $\overline{OS}$ are drawn. If $OR = \frac{2}{3}x - 4$ and $OS = x - 12$, find the length of a diameter of the circle.

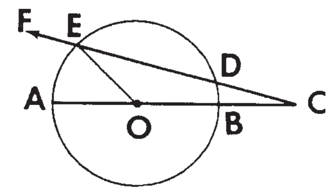
8. In circle O, radii $\overline{OA}$ and $\overline{OB}$ are drawn. If radius $\overline{OC}$ bisects $\angle AOB$, prove $\overline{AC} \cong \overline{BC}$.

9. In circle O, diameter $\overline{AB}$ is drawn. At A, $\overline{CA}$ is drawn perpendicular to $\overline{AB}$. At B, $\overline{DB}$ is drawn perpendicular to $\overline{AB}$. If $\overline{AC} \cong \overline{BD}$, prove $\overline{OC} \cong \overline{OD}$.

10. $\overline{AB}$ is a diameter of circle O, $\overline{AC}$ is any chord, and radius $\overline{OC}$ is drawn. Prove that the bisector of angle BOC is parallel to $\overline{AC}$.

11. On diameter $\overline{AB}$ of semicircle O, two points C and D are located on opposite sides of the center O so that $\overline{AC} \cong \overline{BD}$. At C and D, perpendiculars are erected to $\overline{AB}$ and extended to meet $\overset{\frown}{AB}$ in points E and F respectively. Prove that $\overline{CE} \cong \overline{DF}$.

12. In circle O, diameter $\overline{AB}$ is extended to C. Line $\overleftrightarrow{CF}$ intersects the circle in D and E. If $\overline{DC} \cong \overline{OE}$, show that $m\angle EOA$ is three times as large as $m\angle ACE$. [*Hint:* Draw radius $\overline{OD}$.]



Ex. 12

2. Proving Arcs Equal or Congruent

Theorem 57. **In a circle or in equal circles, central angles whose measures are equal have equal arcs.**

In circle O (Fig. 5–16), if $m\angle AOB = m\angle BOC$, then $\overset{\frown}{AB} = \overset{\frown}{BC}$. We can also say that if $\angle AOB \cong \angle BOC$, then $\overset{\frown}{AB} \cong \overset{\frown}{BC}$.

Corollary T57–1. **A diameter divides a circle into two equal arcs.**

In Fig. 5–17, diameter $\overline{AB}$ divides circle O into two equal arcs, each of which is a semicircle.

Theorem 58. **In a circle or in equal circles, equal arcs have central angles whose measures are equal.**

In circle O (Fig. 5–16), if $\overset{\frown}{AB} = \overset{\frown}{BC}$, then $m\angle AOB = m\angle BOC$. We can also say that if $\overset{\frown}{AB} \cong \overset{\frown}{BC}$, then $\angle AOB \cong \angle BOC$.

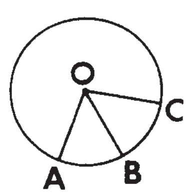


Fig. 5–16

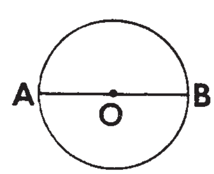


Fig. 5–17

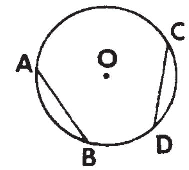


Fig. 5–18

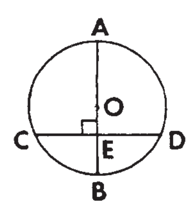


Fig. 5–19

***Theorem 59.* In a circle or in equal circles, congruent chords have equal arcs.**

In circle O (Fig. 5–18), if chord $\overline{AB} \cong$ chord $\overline{CD}$, then $\overset{\frown}{AB} = \overset{\frown}{CD}$. We can also say that: (1) if chord $\overline{AB} \cong$ chord $\overline{CD}$, then $\overset{\frown}{AB} \cong \overset{\frown}{CD}$ and (2) if $AB = CD$, then $\overset{\frown}{AB} = \overset{\frown}{CD}$.

***Theorem 60.* A diameter perpendicular to a chord of a circle bisects the chord and its arcs.**

[The proof for this theorem appears on pages 753–754.]

In circle O (Fig. 5–19), if diameter $\overline{AB} \perp$ chord $\overline{CD}$, then (1) $\overline{CE} \cong \overline{ED}$ or $CE = ED$ (2) $\overset{\frown}{CB} \cong \overset{\frown}{BD}$ or $\overset{\frown}{CB} = \overset{\frown}{BD}$ (3) $\overset{\frown}{CA} \cong \overset{\frown}{AD}$ or $\overset{\frown}{CA} = \overset{\frown}{AD}$.

***Corollary T60–1.* The perpendicular bisector of a chord of a circle passes through the center of the circle.**

In circle O (Fig. 5–19), if $\overleftrightarrow{AB}$ is the perpendicular bisector of chord $\overline{CD}$, then $\overleftrightarrow{AB}$ passes through O, the center of the circle, and $\overline{AB}$ is thus a diameter of the circle.

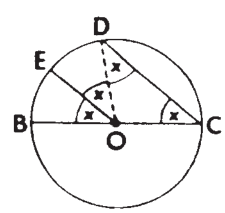
Methods of Proving Arcs in the Same or Equal Circles Equal or Congruent

To prove that arcs in a circle or in equal circles are equal or congruent, prove that either one of the following statements is true:

1. The central angles of the arcs are equal in angular measure, or are congruent.
2. The chords of the arcs are equal in linear measure, or are congruent.

MODEL PROBLEM

Given: In circle O, $\overline{BC}$ is a diameter.
Radius $\overline{OE} \parallel$ chord $\overline{CD}$.



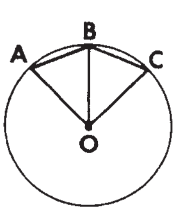
To prove: $\overset{\frown}{BE} = \overset{\frown}{ED}$.

Plan: In order to prove that $\overset{\frown}{BE} = \overset{\frown}{ED}$, we can show that these arcs have central angles whose measures are equal. Draw $\overline{OD}$ to form central angle EOD. Prove $m\angle BOE = m\angle EOD$.

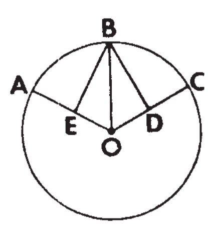
[The proof is given on the next page.]

Proof: *Statements*	*Reasons*
1. Draw radius $\overline{OD}$.	1. One and only one straight line may be drawn through two points.
2. $\overline{OE} \parallel \overline{CD}$.	2. Given.
3. $m\angle EOD = m\angle CDO = x$.	3. If two parallel lines are cut by a transversal, the alternate interior angles are congruent.
4. $\overline{OD} \cong \overline{OC}$.	4. Radii of a circle are congruent.
5. $m\angle OCD = m\angle CDO = x$.	5. If two sides of a triangle are congruent, the angles opposite these sides are congruent.
6. $m\angle BOE = m\angle OCD = x$.	6. If two parallel lines are cut by a transversal, the corresponding angles are congruent.
7. $m\angle BOE = m\angle EOD$.	7. If quantities are equal to the same quantity or equal quantities, they are equal to each other.
8. $\overset{\frown}{BE} = \overset{\frown}{ED}$.	8. In a circle, central angles whose measures are equal have equal arcs.

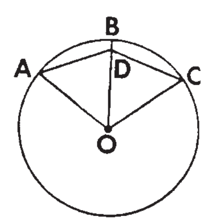
EXERCISES



Ex. 1



Ex. 2–3



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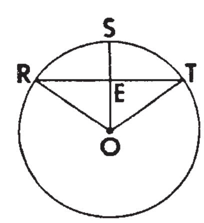


Ex. 7

1. *Given:* In circle O, B is the midpoint of $\overset{\frown}{AC}$.
Prove: $\overline{AB} \cong \overline{BC}$.

2. *Given:* In circle O, $\overline{BE} \perp \overline{OA}$, $\overline{BD} \perp \overline{OC}$, B is the midpoint of $\overset{\frown}{AC}$.
Prove: $\overline{BE} \cong \overline{BD}$.

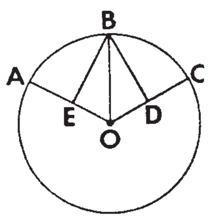
3. *Given:* In circle O, $\overline{BO}$ bisects $\angle EBD$, B is the midpoint of $\overset{\frown}{AC}$.
Prove: $\overline{OE} \cong \overline{OD}$.

4. *Prove:* If an equilateral triangle is inscribed in a circle, it divides the circle into three equal arcs.

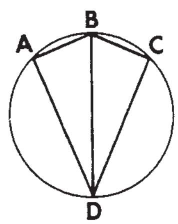
5. If triangle ABC is inscribed in a circle and $\angle A \cong \angle C$, prove $\overset{\frown}{AB} = \overset{\frown}{BC}$.

6. *Given:* In circle O, $\overline{AD} \cong \overline{DC}$.
Prove: $\overset{\frown}{AB} = \overset{\frown}{BC}$.

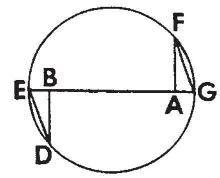
7. *Given:* In circle O, $\overline{OS} \perp \overline{RT}$.
Prove: $\overset{\frown}{RS} \cong \overset{\frown}{ST}$.



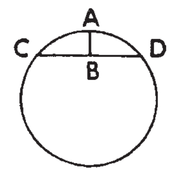
Ex. 8



Ex. 9



Ex. 10



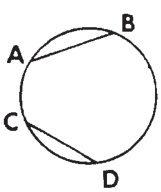
Ex. 11

8. *Given:* In circle O, $\overline{BE} \cong \overline{BD}$, $\overline{BE} \perp \overline{OA}$, $\overline{BD} \perp \overline{OC}$.
Prove: $\overset{\frown}{AB} \cong \overset{\frown}{BC}$.

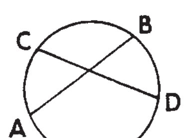
9. *Given:* $\overline{BA} \perp \overline{AD}$, $\overline{BC} \perp \overline{CD}$, $\overline{BD}$ bisects $\angle ABC$.
Prove: $\overset{\frown}{AB} = \overset{\frown}{BC}$.

10. *Given:* $\overline{ED} \parallel \overline{FG}$, $\overline{BD} \cong \overline{AF}$, $\overline{DB} \perp \overline{EG}$, $\overline{FA} \perp \overline{EG}$.
Prove: $\overset{\frown}{DE} \cong \overset{\frown}{FG}$.

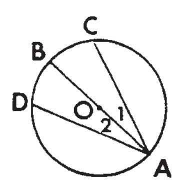
11. $\overline{AB}$ is the perpendicular bisector of $\overline{CD}$. Prove $\overset{\frown}{AC} \cong \overset{\frown}{AD}$.



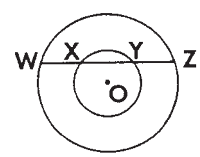
Ex. 12



Ex. 13



Ex. 14–15



Ex. 16

12. If $\overline{AB} \cong \overline{CD}$, prove $\overset{\frown}{BAC} = \overset{\frown}{DCA}$.

13. If $\overline{AB} \cong \overline{CD}$, prove $\overset{\frown}{AC} = \overset{\frown}{BD}$.

14. In circle O, if $\overline{AD} \cong \overline{AC}$ and $\angle 1 \cong \angle 2$, prove $\overset{\frown}{DB} = \overset{\frown}{BC}$.

15. In circle O, if $\overline{AB}$ is a diameter and $\angle 1 \cong \angle 2$, prove $\overset{\frown}{DB} \cong \overset{\frown}{BC}$.

16. *Given:* $\overleftrightarrow{WZ}$ intersects the two concentric circles whose centers are O.
Prove: $\overline{WX} \cong \overline{YZ}$. [*Hint:* Draw $\overline{OM} \perp \overline{WZ}$, M being a point on $\overline{WZ}$.]

17. In a circle, chord $\overline{CD}$ is parallel to diameter $\overline{AB}$. Prove $\overset{\frown}{AC} \cong \overset{\frown}{BD}$.

18. *Prove:* The diagonals of a parallelogram that is inscribed in a circle are congruent.
19. *Prove:* The diagonals of a regular pentagon that is inscribed in a circle are congruent.
20. In circle O, diameter $\overline{AB}$ is perpendicular to chord $\overline{FH}$ at C. If $\overline{FC}$ is 4 inches long, how many inches are there in the length of $\overline{HC}$?
21. In circle O, $\overline{OD}$ is drawn perpendicular to chord $\overline{RS}$, D being a point on $\overline{RS}$. If $RD = 6x + 8$ and $SD = 10x - 36$, find RD and SD.
22. In circle O, $\overline{OC}$ is drawn perpendicular to chord $\overline{AB}$, C being a point on $\overline{AB}$. If $AC = 5x - 1$ and $CB = 13 - 2x$, find the length of chord $\overline{AB}$.

3. Proving Chords Congruent

Theorem 61. **In a circle or in equal circles, equal arcs have congruent chords.**

In circle O (Fig. 5–20), if $\overset{\frown}{AB} = \overset{\frown}{CD}$, then chord $\overline{AB} \cong$ chord $\overline{CD}$. We can also say that if $\overset{\frown}{AB} \cong \overset{\frown}{CD}$, then chord $\overline{AB} \cong$ chord $\overline{CD}$.

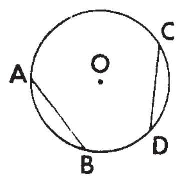


Fig. 5–20

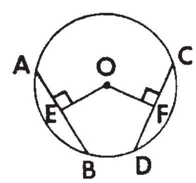


Fig. 5–21

Theorem 62. **In a circle, if two chords are congruent, they are equally distant from the center of that circle.**

In circle O (Fig. 5–21), if chord $\overline{AB} \cong$ chord $\overline{CD}$, $\overline{OE} \perp \overline{AB}$, and $\overline{OF} \perp \overline{CD}$, then $\overline{OE} \cong \overline{OF}$. Hence, $OE = OF$.

Theorem 63. **In a circle, if two chords are equidistant from the center, they are congruent.**

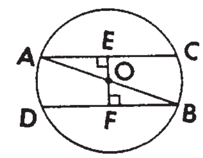
In circle O (Fig. 5–21), if $\overline{OE} \perp \overline{AB}$, $\overline{OF} \perp \overline{CD}$, and $OE = OF$, then chord $\overline{AB} \cong$ chord $\overline{CD}$. Hence, $AB = CD$.

Methods of Proving Chords in the Same or Equal Circles Equal in Length or Congruent

To prove that two chords in a circle or in equal circles are equal in length or congruent, prove that either one of the following statements is true:

1. The arcs of the chords are equal or congruent.
2. The chords are equidistant from the center of the circle.

MODEL PROBLEM



Given: $\overline{AB}$ is a diameter in circle O.
Chord $\overline{AC} \parallel$ chord $\overline{DB}$.

To prove: Chord $\overline{AC} \cong$ chord $\overline{DB}$.

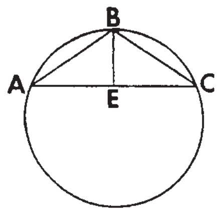
Plan: To prove that chord $\overline{AC}$ and chord $\overline{DB}$ are congruent, show that their distances from O, the center of the circle, are equal. Draw $\overline{OE} \perp \overline{AC}$ and $\overline{OF} \perp \overline{DB}$. Prove $\overline{OE} \cong \overline{OF}$ by proving $\triangle OEA \cong \triangle OFB$ by s.a.a. $\cong$ s.a.a.

Proof:

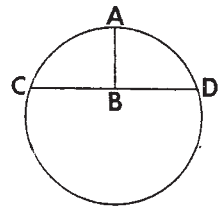
Statements	*Reasons*
1. Draw $\overline{OE} \perp \overline{AC}$, $\overline{OF} \perp \overline{DB}$.	1. From a given point outside a line, one and only one perpendicular can be drawn to the line.
2. $\angle E \cong \angle F$. (a. $\cong$ a.)	2. All right angles are congruent.
3. $\overline{AC} \parallel \overline{DB}$.	3. Given.
4. $\angle EAO \cong \angle FBO$. (a. $\cong$ a.)	4. If two parallel lines are cut by a transversal, the alternate interior angles are congruent.
5. $\overline{OA} \cong \overline{OB}$. (s. $\cong$ s.)	5. Radii of a circle are congruent.
6. $\triangle OEA \cong \triangle OFB$.	6. s.a.a. $\cong$ s.a.a.
7. $\overline{OE} \cong \overline{OF}$, or $OE = OF$.	7. Corresponding parts of congruent triangles are congruent.
8. Chord $\overline{AC} \cong$ chord $\overline{DB}$.	8. In a circle, if two chords are equidistant from the center, they are congruent.

EXERCISES

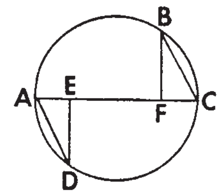
1. If a circle is divided into three equal arcs, and lines are drawn connecting the points of division, prove that an equilateral triangle is formed.



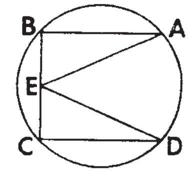
Ex. 2



Ex. 3

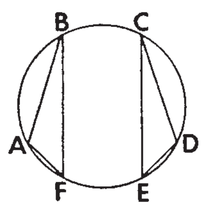


Ex. 4



Ex. 5

2. *Given:* $\overline{BE} \perp \overline{AC}$, B is the midpoint of $\overset{\frown}{AC}$.
   Prove: $\overline{AE} \cong \overline{CE}$.
3. *Given:* B is the midpoint of chord $\overline{CD}$, A is the midpoint of $\overset{\frown}{CD}$.
   Prove: $\overline{AB} \perp \overline{CD}$.
4. *Given:* $\overleftrightarrow{AC}$ is a line, $\overset{\frown}{BC} = \overset{\frown}{AD}$, $\overline{BC} \parallel \overline{AD}$, $\angle B \cong \angle D$.
   Prove: $\overline{AE} \cong \overline{CF}$.
5. *Given:* $\overset{\frown}{AB} \cong \overset{\frown}{CD}$, E is the midpoint of $\overline{BC}$, $\overline{AB} \perp \overline{BC}$, $\overline{DC} \perp \overline{BC}$.
   Prove: $\overline{AE} \cong \overline{DE}$.



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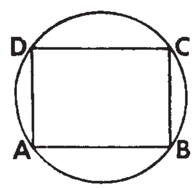

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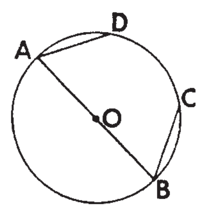
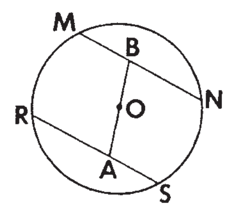
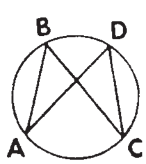


Ex. 8

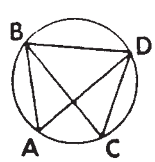


Ex. 9

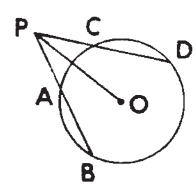
6. *Given:* $\overline{AB} \cong \overline{CD}$, $\overline{AF} \cong \overline{DE}$.
   Prove: $\overline{BF} \cong \overline{CE}$.
7. *Given:* $\overset{\frown}{AB} \cong \overset{\frown}{DC}$, $\overset{\frown}{AD} \cong \overset{\frown}{BC}$.
   Prove: $ABCD$ is a parallelogram.
8. *Given:* In circle O, $\overline{AB}$ is a diameter, $\angle DAB \cong \angle CBA$.
   Prove: $\overline{AD} \cong \overline{BC}$.
9. *Given:* In circle O, $\overline{MN} \parallel \overline{RS}$, $\overline{OB} \cong \overline{OA}$, $\overleftrightarrow{AOB}$ is a line.
   Prove: $\overline{MN} \cong \overline{RS}$.



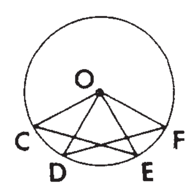
Ex. 10



Ex. 11



Ex. 12



Ex. 13

10. If chord $\overline{AD}$ is congruent to chord $\overline{CB}$, prove that chord $\overline{AB}$ is congruent to chord $\overline{CD}$.

11. If chord $\overline{AB}$ is congruent to chord $\overline{CD}$, prove that $\triangle DAB \cong \triangle BCD$.
12. In circle O, $\overline{PD}$ and $\overline{PB}$ form congruent angles with $\overline{PO}$. Prove that $\overline{AB} \cong \overline{CD}$.
13. In circle O, if $\angle COD \cong \angle FOE$, prove that $\overline{CE} \cong \overline{DF}$.
14. A is a point on circle O. Radius $\overline{OA}$ is drawn. Chords $\overline{AB}$ and $\overline{AC}$ are drawn on opposite sides of $\overline{OA}$ so that $\angle BAO \cong \angle CAO$. Prove that chord $\overline{AB}$ is congruent to chord $\overline{AC}$.
15. If two chords which intersect on a circle make congruent angles with the radius drawn to the point of intersection, prove that the chords are congruent.
16. *Prove:* The line that bisects the minor arc and the major arc of a chord is the perpendicular bisector of the chord.
17. *Prove:* If equilateral triangles are inscribed in two congruent circles, the triangles are congruent.
18. *Prove:* In a circle, two chords perpendicular to a third chord at its end-points are congruent.
19. *Prove:* A radius of a circle which bisects an arc in the circle is the perpendicular bisector of the chord of that arc.
20. In circle O, $\overparen{AB} = \overparen{CD}$. If $AB = 9x + 10$ and $CD = 4x + 60$, find AB.
21. In circle O, chord $\overline{LM} \cong$ chord $\overline{NP}$. If $\overparen{LM} = \frac{2}{3}y + 20$ and $\overparen{NP} = 2y - 20$, find $\overparen{NP}$.
22. Chords $\overline{AB}$ and $\overline{CD}$ are drawn in circle O. $\overline{OE} \perp \overline{AB}$, and $\overline{OF} \perp \overline{CD}$, point E being on $\overline{AB}$ and point F being on $\overline{CD}$.
    a. If $\overline{AB} \cong \overline{CD}$, $OE = 5x + 1$, and $OF = 8x - 11$, find OE.
    b. If $\overline{OE} \cong \overline{OF}$, $AB = 3x + 7$, and $CD = 5x - 7$, find AE.
23. Use an indirect method of proof to prove: In the same or in equal circles, chords that are not congruent are unequally distant from the center.

4. Tangents and Tangent Circles

A Tangent to a Circle

***Definition.* A *tangent to a circle* is a line in the plane of the circle which intersects the circle at one and only one point. The point at which the tangent intersects the circle is called the "point of tangency," or "point of contact."**

In Fig. 5–22 on the next page, $\overleftrightarrow{AB}$ is a tangent to circle O; and point P is the point of tangency, or point of contact.

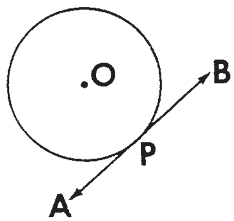


Fig. 5–22

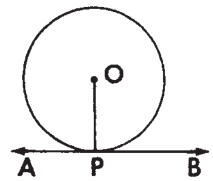


Fig. 5–23

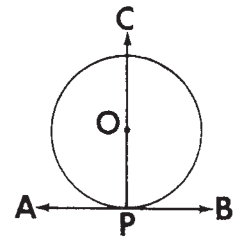


Fig. 5–24

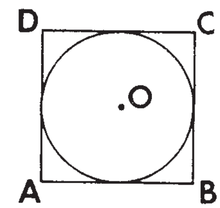


Fig. 5–25

***Theorem 64.* If a line is tangent to a circle, the line is perpendicular to the radius drawn to the point of contact.**

In Fig. 5–23, if $\overleftrightarrow{AB}$ is tangent to circle O, then tangent $\overleftrightarrow{AB} \perp$ radius $\overline{OP}$ at P.

***Theorem 65.* If a line is perpendicular to a radius of a circle at its outer endpoint, then the line is a tangent to the circle.**

In circle O (Fig. 5–23), if $\overleftrightarrow{AB} \perp$ radius $\overline{OP}$ at P, then $\overleftrightarrow{AB}$ is tangent to circle O.

***Theorem 66.* If a line is perpendicular to a tangent to a circle at the point of contact, the line passes through the center of the circle.**

In Fig. 5–24, if $\overleftrightarrow{AB}$ is tangent to circle O at point P and $\overleftrightarrow{CP} \perp \overleftrightarrow{AB}$ at P, then $\overleftrightarrow{CP}$ passes through the center of circle O.

***Definition.* A *circle is inscribed in a polygon* when all the sides of the polygon are tangent to the circle.**

In Fig. 5–25, if all sides of polygon $ABCD$ are tangent to circle O, then circle O is inscribed in polygon $ABCD$. We can also say that polygon $ABCD$ is "circumscribed about circle O."

Tangents to a Circle From an External Point

***Definition.* The *length of a tangent from an external point to a circle* is the length of the line segment whose endpoints are the external point and the point of contact of the tangent to the circle.**

In Fig. 5–26, if $\overleftrightarrow{AB}$ is tangent to circle O at point P, the length of the tangent from point A to circle O is the length of line segment $\overline{AP}$. Note that this length does not measure the length of tangent $\overleftrightarrow{AB}$. Since $\overleftrightarrow{AB}$ is a line and not a line segment, $\overleftrightarrow{AB}$ has no length. For convenience, segment $\overline{AP}$ may be referred to as the tangent from A. Segment $\overline{AP}$ may also be referred to as a *tangent segment*.

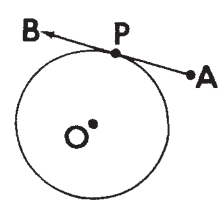


Fig. 5–26

Theorem 67. **If two tangents are drawn to a circle from an external point, these tangents are equal in length.**

In Fig. 5–27, if $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangent to circle O, then $PA = PB$.

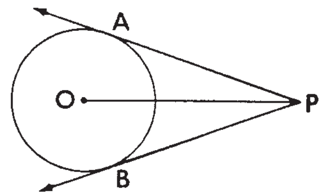


Fig. 5–27

Corollary T67–1. **If two tangents are drawn to a circle from an external point, the line passing through that point and the center of the circle bisects the angle formed by the tangents.**

In Fig. 5–27, if $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangents drawn to circle O, then $\overleftrightarrow{PO}$ bisects $\angle APB$, or $\angle APO \cong \angle BPO$.

Line of Centers of Two Circles

Definition. **The *line of centers of two circles* is the line segment whose endpoints are the centers of the circles.**

In Fig. 5–28, $\overline{OO'}$ is the line of centers of circles O and O'.

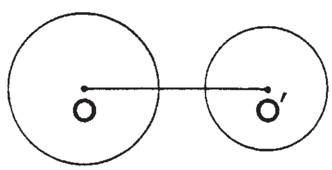


Fig. 5–28

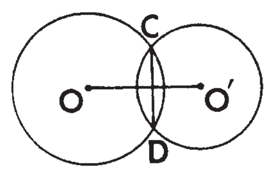


Fig. 5–29

Theorem 68. **If two circles intersect in two points, their line of centers is the perpendicular bisector of their common chord.**

If circles O and O' intersect in point C and point D (Fig. 5–29), their line of centers $\overline{OO'}$ is the perpendicular bisector of common chord $\overline{CD}$. When two circles intersect in two points, their *common chord* is the line segment whose endpoints are the two points of intersection.

Common Tangents

Definition. **A *common tangent* to two circles is a line which is tangent to each of the circles.**

On the next page are shown some examples of common tangents drawn to circles:

Common Internal Tangents

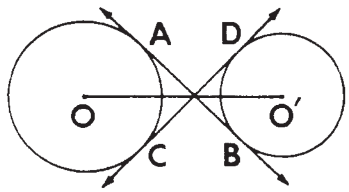


Fig. 5–30

Common External Tangents

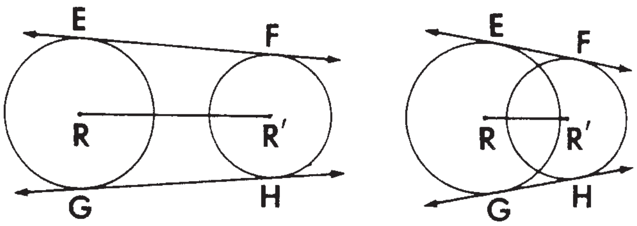


Fig. 5–31

***Definition.* A *common internal tangent* to two circles is a line which is tangent to both circles and intersects their line of centers.**

In Fig. 5–30, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are common internal tangents to circles O and O'. Both $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect $\overline{OO'}$, the line of centers. The length of common internal tangent $\overleftrightarrow{AB}$ is the length of line segment $\overline{AB}$, whose endpoints are the points of contact A and B.

***Definition.* A *common external tangent* to two circles is a line which is tangent to both circles and does not intersect their line of centers.**

In Fig. 5–31, $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are common external tangents to circles R and R'. Neither $\overleftrightarrow{EF}$ nor $\overleftrightarrow{GH}$ intersects $\overline{RR'}$, the line of centers. The length of common external tangent $\overleftrightarrow{EF}$ is the length of line segment $\overline{EF}$, whose endpoints are the points of contact E and F.

Tangent Circles

***Definition. Tangent circles* are circles in a plane that are tangent to the same line at the same point.**

Following are examples of tangent circles:

Internally Tangent Circles

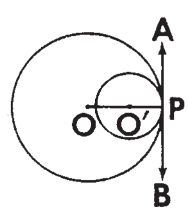


Fig. 5–32

Externally Tangent Circles

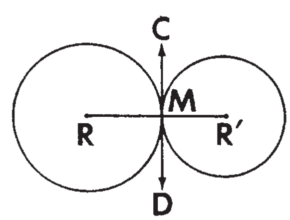


Fig. 5–33

***Definition. Internally tangent circles* are tangent circles which lie on the same side of the common tangent.**

In Fig. 5–32, circles O and O' are internally tangent because they both lie to the left of common tangent $\overleftrightarrow{AB}$.

Definition. Externally tangent circles **are tangent circles which lie on opposite sides of the common tangent.**

In Fig. 5–33, circles R and R' are externally tangent because circle R lies to the left of common tangent $\overleftrightarrow{CD}$, and circle R' lies to the right of $\overleftrightarrow{CD}$.

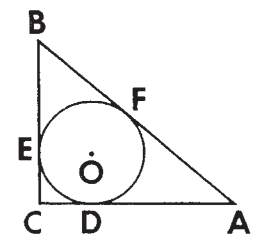
Theorem 69. **If two circles are tangent, their line of centers, extended if necessary, passes through the point of contact and is perpendicular to their common tangent.**

In Fig. 5–32, the line of centers $\overline{OO'}$, extended, passes through the point of contact P and is perpendicular to common tangent $\overleftrightarrow{AB}$.

Also, in Fig. 5–33, the line of centers RR' passes through the point of contact M and is perpendicular to common tangent $\overleftrightarrow{CD}$.

MODEL PROBLEMS

1. $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{CA}$ are tangents to circle O. $AD = 5$ and $BE = 4$. Find the length of $\overline{AB}$.



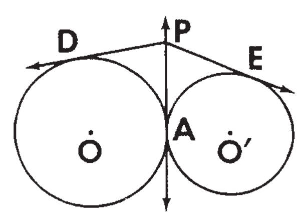
Solution: Since the lengths of the tangents drawn to a circle from an external point are equal,

$$AF = AD = 5 \quad \text{and} \quad BF = BE = 4$$

Therefore, $AB = AF + BF = 5 + 4 = 9$.

Answer: $AB = 9$.

2. Prove that if two circles are tangent externally, tangents drawn to the circles from any point in their common internal tangent are equal in length.



Given: Circles O and O' are externally tangent.
$\overleftrightarrow{PA}$ is their common internal tangent.
$\overleftrightarrow{PD}$ is tangent to circle O.
$\overleftrightarrow{PE}$ is tangent to circle O'.

To prove: $PD = PE$.

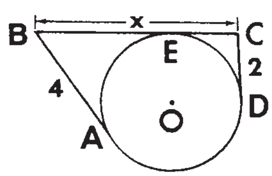
Plan: To prove that the length of tangent $\overleftrightarrow{PD}$ is equal to the length of tangent $\overleftrightarrow{PE}$, prove that both PD and PE are equal to PA, the length of tangent $\overleftrightarrow{PA}$.

[The proof is given on the next page.]

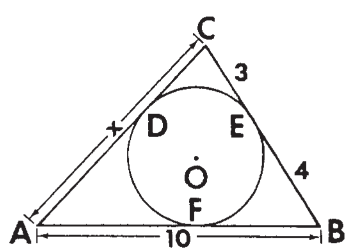
Proof: *Statements*	*Reasons*
1. $\overleftrightarrow{PD}$ and $\overleftrightarrow{PA}$ are tangents to circle O.	1. Given.
2. $PD = PA$.	2. The lengths of tangents drawn to a circle from an external point are equal.
3. $\overleftrightarrow{PE}$ and $\overleftrightarrow{PA}$ are tangents to circle O'.	3. Given.
4. $PA = PE$.	4. The lengths of tangents drawn to a circle from an external point are equal.
5. $PD = PE$.	5. Transitive property of equality.

EXERCISES

In 1–3, find the value of x, or x and y, as indicated.



1.



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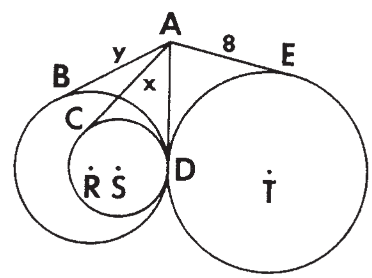


3.

4. $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangents to circle O from point P. Chord $\overline{AB}$ is drawn. Find the number of degrees contained in $\angle PAB$ if $\angle APB$ contains:
a. 80° *b.* 40° *c.* 60° *d.* 90° *e.* 120°

5. $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangents to circle O from point P, and chord $\overline{AB}$ is drawn. Express the number of degrees contained in angle PAB if the number of degrees contained in angle APB is represented by:
a. x *b.* $2m$ *c.* $(180 - x)$ *d.* $(90 - x)$ *e.* $(x + y)$

6. $\overleftrightarrow{PC}$ and $\overleftrightarrow{PD}$ are tangents to circle O at C and D respectively. $\overline{OC}$ and $\overline{OD}$ are drawn. Find $m\angle CPD$ if $m\angle COD$ is:
a. 160 *b.* 140 *c.* 120 *d.* 90 *e.* 60 *f.* x *g.* $(180 - x)$

7. Prove that tangents drawn to a circle at the ends of a diameter are parallel.

8. *Prove:* Tangents to a circle from a point outside the circle form congruent angles with the chord joining their points of contact.
9. *Prove:* If two circles are tangent externally, their common internal tangent bisects their common external tangents.
10. *Prove:* If two nonintersecting circles, each outside the other, are congruent, their line of centers bisects a common internal tangent.
11. *Prove:* The common internal tangents to two nonintersecting circles, each outside the other, are congruent.
12. *Prove:* The common external tangents to two unequal nonintersecting circles, each outside the other, are congruent.
13. Points C and D are on circle O. Tangents drawn to circle O at points C and D intersect at P. Prove that $\overline{PO}$ bisects minor arc $\widehat{CD}$.
14. *Prove:* If two tangents drawn to a circle from an external point meet at an angle of 60°, the chord joining their points of contact is congruent to each tangent.
15. *Prove:* An angle formed by two tangents drawn to a circle from an external point is supplementary to the angle formed by the radii drawn to the points of contact.
16. *Prove:* The sum of the lengths of two opposite sides of a quadrilateral that is circumscribed about a circle is equal to the sum of the lengths of the other two sides.
17. *Prove:* If two circles are tangent externally, the common internal tangent bisects a common external tangent.
18. *Prove:* If a circle is inscribed in a right triangle, the sum of the length of the hypotenuse of the triangle and the length of the diameter of the circle is equal to the sum of the lengths of the legs of the triangle.
19. Draw two circles which will have the indicated number of common tangents. (*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3 (*e*) 4
20. Draw two circles whose radii are 3 and 4 if the length of the line of centers is:
    a. 10 *b.* 6 *c.* 7 *d.* 1
21. State the number of common tangents that can be drawn to: (*a*) two circles which intersect in two points (*b*) two circles, each outside the other, that do not intersect (*c*) two circles that are externally tangent (*d*) two circles that are internally tangent.
22. If the radius of circle O is represented by r, the radius of circle O' is represented by R, and the length of the line of centers OO' is represented by D, express the relationship among D, r, and R when (*a*) the two circles are tangent externally (*b*) the two circles are tangent internally (*c*) the two circles intersect in two points (*d*) the two circles are nonintersecting, with each circle outside the other.

23. *Prove:* If a tangent to a circle is parallel to a chord of a circle, the lines intercept congruent arcs on the circle.

24. *Prove:* Two parallel chords intercept congruent arcs on a circle.

25. *Prove:* Two parallel tangents intercept congruent arcs on a circle.

26. *Prove:* A chord of a circle is parallel to a tangent to the circle drawn at the midpoint of the minor arc of the chord.

27. $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangents drawn to circle O from point P. If $PA = 8r - 7$ and $PB = 2r + 35$, find r, PA, and PB.

28. $\overleftrightarrow{PC}$ and $\overleftrightarrow{PD}$ are tangents drawn to circle O from point P. If $m\angle DOC = 2x + 40$ and $m\angle CPD = x - 10$, find $m\angle DOC$, $m\angle CPD$, and $m\angle CPO$.

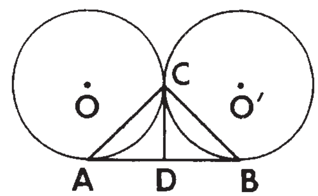
29. $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangents drawn to circle O from point P. If $m\angle APB = 60$, $AP = 2x - 2y$, $PB = \frac{1}{3}x + 6$, and $AB = x + y$, find x, y, and AP.

30. *Given:* $\overleftrightarrow{CD}$ is a common internal tangent to circles O and O'. $\overline{AB}$ is a common external tangent to circles O and O'.

Prove: *a.* D is the midpoint of $\overline{AB}$.

b. $\angle ACB$ is a right angle.

[*Hint:* Let $m\angle BCD = x$; let $m\angle ACD = y$.]



Ex. 30

5. Measurements of Angles and Arcs

Measuring a Central Angle

We have learned that in a circle a central angle is an angle whose vertex is the center of the circle. Also, we have defined the measure of a minor arc as the measure of the central angle that intercepts the arc. Since a definition is reversible we can also say that:

The measure of a central angle is the measure of its intercepted arc.

In circle O (Fig. 5–34), if $\angle AOB$ is a central angle, then $m\angle AOB = m\widehat{AB}$, or $m\angle AOB = b$.

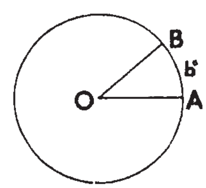


Fig. 5–34

***Theorem 70.* In a circle or in equal circles, two arcs which contain the same number of arc degrees are equal.**

In equal circles O and O' (Fig. 5–35), if the number of degrees in $\widehat{AB}$ is equal to the number of degrees in $\widehat{A'B'}$, then $\widehat{AB} = \widehat{A'B'}$.

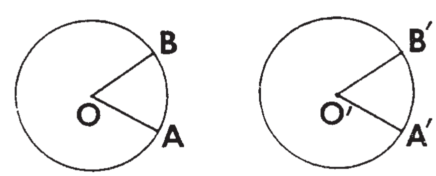


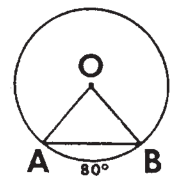
Fig. 5–35

Method of Proving Angles Congruent Using Central Angles

To prove that two angles are congruent or equal in measure, prove that they are central angles in a circle or in equal circles and that they intercept the same or equal arcs.

MODEL PROBLEM

In circle O, $m\widehat{AB} = 80$. Find the measure of $\angle A$.



Solution: Since $\angle AOB$ is a central angle, $m\angle AOB = m\widehat{AB}$, or $m\angle AOB = 80$.
Since radius $\overline{OA} \cong$ radius $\overline{OB}$, $m\angle A = m\angle B$.

1. $m\angle A + m\angle B + m\angle AOB = 180$
2. $m\angle A + m\angle A + 80 = 180$
3. $2m\angle A + 80 = 180$
4. $2m\angle A = 100$
5. $m\angle A = 50$

Answer: $m\angle A = 50$.

EXERCISES

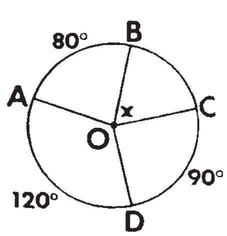
1. Find the measure of a central angle which intercepts an arc whose measure is:
a. 50 *b.* 20 *c.* 70 *d.* 90 *e.* r *f.* $4x$

2. Find the measure of the arc intercepted by a central angle whose measure is:
a. 30 *b.* 60 *c.* 75 *d.* 120 *e.* b *f.* $180 - x$

3. A central angle whose measure is 40 intercepts an arc whose measure is ______ degrees.

4. Angle AOB is a central angle of circle O, and chord $\overline{AB}$ is drawn. If the length of chord $\overline{AB}$ is equal to the length of a radius, find the measure of angle AOB.

In 5–9, O is the center of each circle. Find the value of x.



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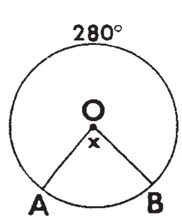

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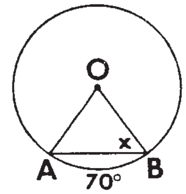

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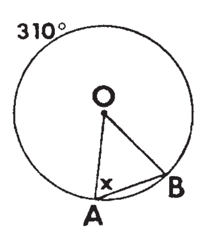

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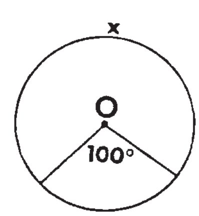


9.

10. As the number of degrees in an arc of a circle increases, what change takes place in its central angle?
11. If the measure of an arc which is less than 180 is doubled, what change takes place in the measure of its central angle?
12. If the measure of an arc is halved, what change takes place in the measure of its central angle?

Measuring an Inscribed Angle

***Definition.* In a circle, an *inscribed angle* is an angle whose vertex lies on the circle and whose sides are chords of the circle.**

In circle O (Fig. 5–36), $\angle ACB$ is called an inscribed angle and is said to be inscribed in the circle.

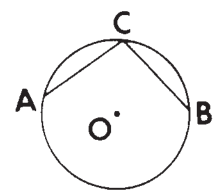


Fig. 5–36

***Definition.* An *angle is inscribed in an arc* of a circle if its vertex lies on the arc and its sides are chords which join the vertex and the ends of the arc.**

In Fig. 5–36, $\angle ACB$ is inscribed in $\widehat{ACB}$. Angle ACB *intercepts,* or cuts off, $\widehat{AB}$.

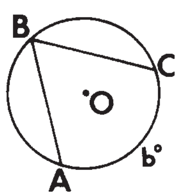


Fig. 5–37

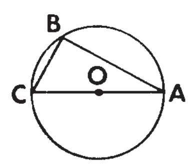


Fig. 5–38

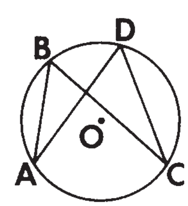


Fig. 5–39

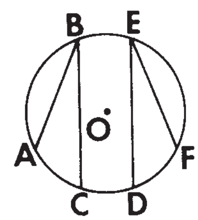


Fig. 5–40

***Theorem 71.* The measure of an angle inscribed in a circle is equal to one-half the measure of its intercepted arc.**

[The proof for this theorem appears on pages 755–757.]

In circle O (Fig. 5–37), $m\angle ABC = \frac{1}{2}m\widehat{AC}$, or $m\angle ABC = \frac{1}{2}b$.

***Corollary T71–1.* An angle inscribed in a semicircle is a right angle.**

In Fig. 5–38, if $\overline{AC}$ is a diameter in circle O, angle CBA is a right angle.

***Corollary T71–2.* In a circle or in equal circles, if inscribed angles intercept the same or equal arcs, then the inscribed angles are equal in measure.**

In circle O (Fig. 5–39), $m\angle ABC$, which intercepts $\widehat{AC}$, and $m\angle ADC$, which also intercepts $\widehat{AC}$, are equal, or $m\angle ABC = m\angle ADC$.

***Corollary T71–3.* In a circle or in equal circles, if inscribed angles are equal in measure, then they intercept equal arcs.**

In circle O (Fig. 5–40), if $m\angle ABC = m\angle DEF$, then $\widehat{AC} = \widehat{DF}$.

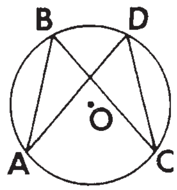


Fig. 5–41

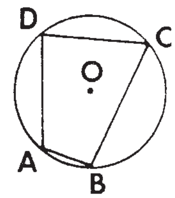


Fig. 5–42

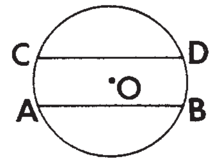


Fig. 5–43

***Corollary T71–4.* In a circle, angles inscribed in the same arc are equal in measure.**

In circle O (Fig. 5–41), $\angle ABC$, which is inscribed in $\widehat{ABDC}$, is equal in measure to $\angle ADC$, which is also inscribed in $\widehat{ABDC}$, or $m\angle ABC = m\angle ADC$.

***Corollary T71–5.* In a circle, the opposite angles of an inscribed quadrilateral are supplementary.**

In Fig. 5–42, if quadrilateral $ABCD$ is inscribed in circle O, then $m\angle A + m\angle C = 180$ and $m\angle B + m\angle D = 180$. If $m\angle A = x$, then $m\angle C = 180 - x$.

***Corollary T71–6.* Parallel lines which intersect a circle intercept equal arcs on a circle.**

In circle O (Fig. 5–43), if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, then $\widehat{CA} = \widehat{DB}$.

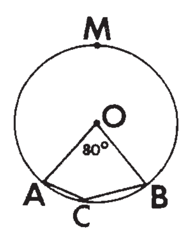
Method of Proving Angles Congruent Using Inscribed Angles

To prove that two angles are congruent or equal in measure, prove that they are inscribed angles in the same circle or in equal circles and that they intercept, or are inscribed in, the same arc or equal arcs.

MODEL PROBLEM

In circle O, the measure of central angle AOB is 80. If point C is on minor arc $\widehat{AB}$, find the measure of $\angle ACB$.

Solution:



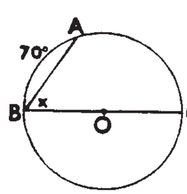
1. Since $m\angle AOB = 80$, the measure of intercepted $\widehat{AB} = 80$.
2. Since the measure of minor $\widehat{AB} = 80$, the measure of major $\widehat{AMB} = 360 - 80$, or 280.
3. The measure of inscribed $\angle ACB$ is equal to one-half the measure of major arc $\widehat{AMB}$.
4. $m\angle ACB = \frac{1}{2} m\widehat{AMB}$
5. $m\angle ACB = \frac{1}{2}(280) = 140$

Answer: The measure of $\angle ACB$ is 140.

EXERCISES

1. Find the measure of an inscribed angle which intercepts an arc whose measure is:
   a. 40 *b.* 80 *c.* 105 *d.* 120 *e.* a *f.* $4y$
2. Find the measure of the arc intercepted by an inscribed angle whose measure is:
   a. 30 *b.* 75 *c.* $22\frac{1}{2}$ *d.* 45 *e.* b *f.* $6a$
3. Inscribed angle ABC of circle O measures 75. Radii $\overline{AO}$ and $\overline{CO}$ are drawn. Find the measure of angle AOC.
4. $\overline{AB}$ is a diameter of a circle, $\overline{AC}$ is a chord, and arc $\widehat{AC}$ contains 100°. Angle BAC contains ______ degrees.
5. If the vertices of an inscribed triangle divide the circle into three arcs whose measures are in the ratio 3:4:5, what is the measure of the largest angle of the triangle?
6. An inscribed angle and a central angle intercept the same arc of a circle. The ratio of the measure of the inscribed angle to the measure of the central angle is ______.
7. In circle O, central angle DOE measures 80. Find the measure of angle DFE, which is inscribed in minor arc DE.
8. Central angle AOB in circle O measures 120, and X is any point on minor arc $\widehat{AB}$. Find the measure of angle AXB.

In 9–13, O is the center of each circle. Find the value of x, or x and y.



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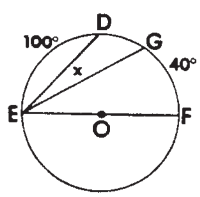

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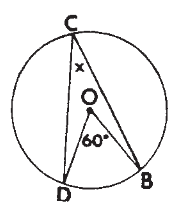

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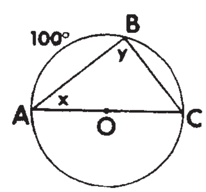

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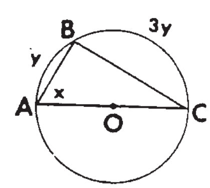
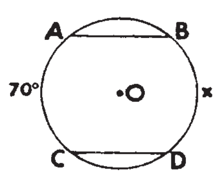


13.

14. In a circle, two parallel chords on opposite sides of the center have arcs which measure 100 and 120. Find the measure of one of the arcs included between the chords.

In 15–19, O is the center of the circle and $\overline{AB} \parallel \overline{CD}$. Find the value of x.



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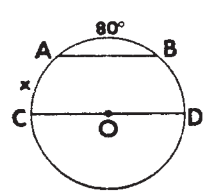

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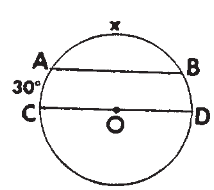

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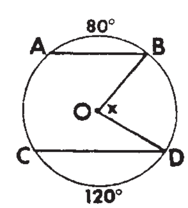

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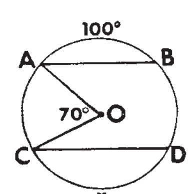


19.

20. If quadrilateral $ABCD$ is inscribed in a circle, find the measure of angle B if its opposite angle D measures:
$a.$ 65 $b.$ 90 $c.$ 110 $d.$ a $e.$ $180 - 2x$
21. Quadrilateral $ABCD$ is inscribed in a circle. If angle A measures 95, find the measure of angle C.
22. Quadrilateral $ABCD$ is inscribed in a circle. If $m\widehat{AB} = 119$, $m\widehat{BC} = 73$, and $m\widehat{CD} = 60$, what is the measure of angle ABC?
23. Prove that when a circle whose diameter is one of the congruent sides of an isosceles triangle is drawn, the circle bisects the base of the triangle.

Measuring an Angle Formed by a Tangent and a Chord

Theorem 72. **The measure of an angle formed by a tangent and a chord drawn from the point of contact is equal to one-half the measure of its intercepted arc.**

In circle O (Fig. 5–44), the measure of $\angle ABC$, which is formed by tangent $\overleftrightarrow{AB}$ and chord $\overline{BC}$, is equal to one-half the measure of its intercepted arc $\widehat{BC}$. $m\angle ABC = \frac{1}{2}m\widehat{BC}$, or $m\angle ABC = \frac{1}{2}b$.

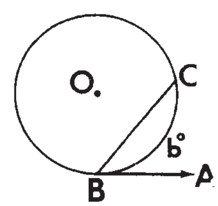
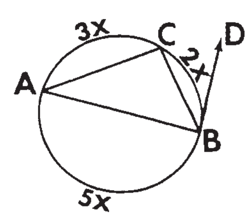


Fig. 5–44

MODEL PROBLEM

In the figure shown, triangle ABC is inscribed in the circle. $m\widehat{BC}:m\widehat{CA}:m\widehat{AB} = 2:3:5$. Find the measure of the acute angle formed by side $\overline{BC}$ and the tangent to the circle at B.



Solution: Represent the measures of arcs $\widehat{BC}$, $\widehat{CA}$, and $\widehat{AB}$ by $2x$, $3x$, and $5x$ respectively.

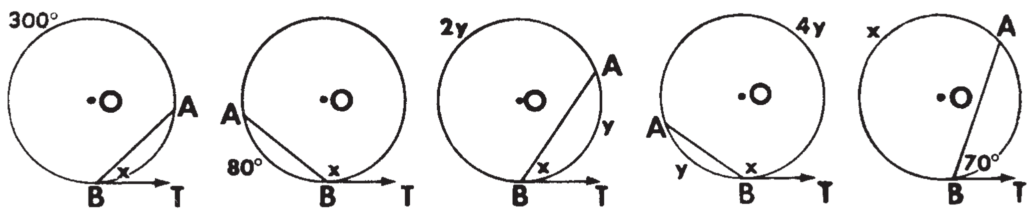
1. $2x + 3x + 5x = 360$
2. $10x = 360$
3. $x = 36$
4. $2x = 72$
5. $m\angle DBC = \frac{1}{2}m\widehat{BC}$, or $m\angle DBC = \frac{1}{2}(72) = 36$.

Answer: $m\angle DBC = 36$.

EXERCISES

1. Find the measure of the angle formed by a tangent and a chord drawn from the point of contact if the angle intercepts an arc whose measure is:
a. 40 *b.* 80 *c.* 65 *d.* 140 *e.* b *f.* $180 - 2x$
2. Find the measure of the arc intercepted by an angle formed by a tangent and a chord drawn from the point of contact if the angle measures:
a. 30 *b.* 50 *c.* $22\frac{1}{2}$ *d.* 70 *e.* m *f.* $\frac{3}{2}x$
3. Triangle ABC is inscribed in a circle, side $\overline{AB}$ is a diameter of the circle, and arc $\widehat{AC}$ measures 100. The measure of the acute angle formed by the tangent at B and side $\overline{BC}$ of the triangle is ______.
4. Equilateral triangle ABC is inscribed in a circle. Find the measure of the acute angle formed by side $\overline{AB}$ and the tangent at B.
5. A regular pentagon $ABCDE$ is inscribed in a circle. Find the measure of the acute angle formed by side $\overline{AB}$ and the tangent at B.
6. Triangle ABC is inscribed in circle O. $m\widehat{AB}:m\widehat{BC}:m\widehat{CA} = 3:3:4$. Find the measure of the acute angle formed by side $\overline{AB}$ and the tangent to the circle at A.

In 7–11, O is the center of the circle and $\overleftrightarrow{BT}$ is tangent to the circle. Find the value of x, or x and y, as indicated.



7. 8. 9. 10. 11.

Measuring an Angle Formed by Two Chords Intersecting Within a Circle

***Theorem 73.* The measure of an angle formed by two chords intersecting within a circle is equal to one-half the sum of the measures of the intercepted arcs.**

[The proof for this theorem appears on page 758.]

In circle O (Fig. 5–45), if chords $\overline{AB}$ and $\overline{CD}$ intersect at E, then the measure of $\angle CEA$ is equal to one-half the sum of the measures of arcs $\widehat{AC}$ and $\widehat{BD}$. $m\angle CEA = \frac{1}{2}(m\widehat{AC} + m\widehat{BD})$, or $m\angle CEA = \frac{1}{2}(a + b)$.

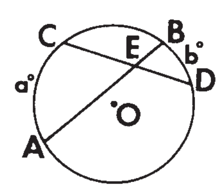
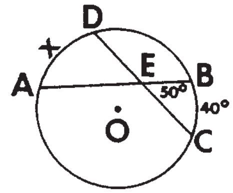


Fig. 5–45

MODEL PROBLEM

In circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect at E. If $m\angle CEB = 50$ and $m\overset{\frown}{CB} = 40$, find the measure of minor arc $\overset{\frown}{AD}$.

Solution: Let $x =$ the measure of minor arc $\overset{\frown}{AD}$.



1. $m\angle CEB = \frac{1}{2}(m\overset{\frown}{BC} + m\overset{\frown}{AD})$
2. $50 = \frac{1}{2}(40 + x)$ Multiply by 2.
3. $100 = 40 + x$
4. $60 = x$

Answer: The measure of minor arc $\overset{\frown}{AD}$ is 60.

EXERCISES

1. Find the measure of the angle formed by two chords intersecting within a circle if the opposite arcs they intercept measure:
a. 40 and 60 *b.* 90 and 20 *c.* 110 and 44 *d.* $4x$ and $6x$

2. Find the sum of the measures of the opposite arcs intercepted by two chords which meet inside a circle if the chords form an angle whose measure is:
a. 50 *b.* 70 *c.* 90 *d.* $37\frac{1}{2}$ *e.* $67\frac{1}{2}$ *f.* $a + b$

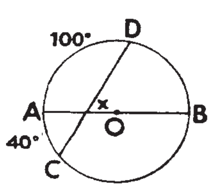
3. Chords $\overline{AB}$ and $\overline{CD}$ of a circle intersect at point E within the circle. If $m\overset{\frown}{AC} = 120$ and $m\overset{\frown}{BD} = 80$, find the measure of angle CEB.

4. Regular pentagon $ABCDE$ is inscribed in circle O. Diagonals $\overline{AD}$ and $\overline{EB}$ intersect at F. Find the measure of angle BFD.

5. Two chords intersecting within a circle form an angle whose measure is 60. If one of the intercepted arcs measures 80, what is the measure of the other intercepted arc?

6. In a circle, chords $\overline{AB}$ and $\overline{CD}$ are perpendicular, and they intersect at E. If $m\overset{\frown}{AC} = 80$, find the measure of arc $\overset{\frown}{BD}$.

In 7–11, O is the center of the circle. Chords $\overline{AB}$ and $\overline{CD}$ intersect inside circle O. Find the value of x, or x and y, as indicated.



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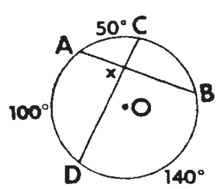

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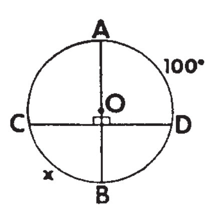

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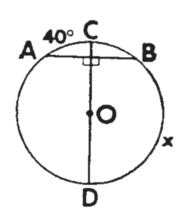

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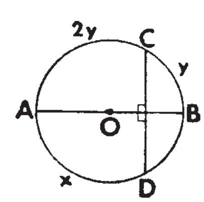


11.

Measuring Angles Formed by Secants and Tangents Drawn to a Circle From an Outside Point

***Definition.* A *secant* is a line which intersects a circle in two points.**

In circle O (Fig. 5–46), $\overleftrightarrow{EF}$ is a secant.

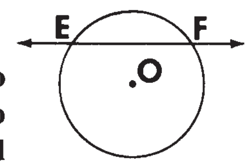


Fig. 5–46

***Theorem 74.* The measure of an angle formed by two secants drawn to a circle from an outside point is equal to one-half the difference of the measures of the intercepted arcs.**

[The proof for this theorem appears on page 759.]

In circle O (Fig. 5–47), if secants $\overleftrightarrow{AC}$ and $\overleftrightarrow{AE}$ are drawn from point A, then the measure of $\angle CAE$ is equal to one-half the difference of the measures of the intercepted arcs $\overset{\frown}{CE}$ and $\overset{\frown}{BD}$. $m\angle CAE = \frac{1}{2}(m\overset{\frown}{CE} - m\overset{\frown}{BD})$, or $m\angle CAE = \frac{1}{2}(a - b)$.

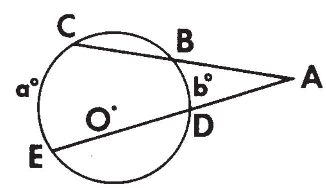


Fig. 5–47

***Theorem 75.* The measure of an angle formed by a tangent and a secant drawn to a circle from an outside point is equal to one-half the difference of the measures of the intercepted arcs.**

[The proof for this theorem appears on page 760.]

In circle O (Fig. 5–48), if tangent $\overleftrightarrow{AB}$ and secant $\overleftrightarrow{AD}$ are drawn from point A, then the measure of $\angle BAD$ is equal to one-half the difference of the measures of intercepted arcs $\overset{\frown}{BD}$ and $\overset{\frown}{BC}$. $m\angle BAD = \frac{1}{2}(m\overset{\frown}{BD} - m\overset{\frown}{BC})$, or $m\angle BAD = \frac{1}{2}(a - b)$.

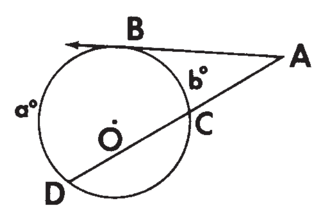


Fig. 5–48

***Theorem 76.* The measure of an angle formed by two tangents drawn to a circle from an outside point is equal to one-half the difference of the measures of the intercepted arcs.**

[The proof for this theorem appears on page 761.]

In circle O (Fig. 5–49), if tangents $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are drawn from point A, then the measure of $\angle BAC$ is equal to one-half the difference of the measures of the intercepted arcs, major arc $\overset{\frown}{BC}$ and minor arc $\overset{\frown}{BC}$. $m\angle BAC = \frac{1}{2}(m \text{ major } \overset{\frown}{BC} - m \text{ minor } \overset{\frown}{BC})$, or $m\angle BAC = \frac{1}{2}(a - b)$.

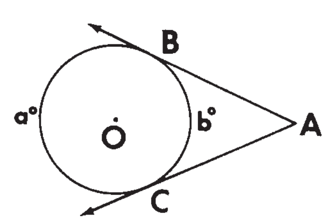
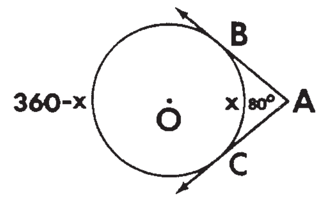


Fig. 5–49

MODEL PROBLEM

The angle formed by two tangents drawn to a circle from the same external point measures 80. Find the measure of the smaller of the intercepted arcs.

Solution: Let x = the measure of minor $\overset{\frown}{BC}$.
Then $360 - x$ = the measure of major $\overset{\frown}{BC}$.



1. $m\angle BAC = \frac{1}{2}(m \text{ major } \overset{\frown}{BC} - m \text{ minor } \overset{\frown}{BC})$
2. $80 = \frac{1}{2}(360 - x - x)$
3. $80 = \frac{1}{2}(360 - 2x)$
4. $80 = 180 - x$
5. $x = 180 - 80$
6. $x = 100$

Answer: The measure of the smaller intercepted arc is 100.

NOTE. Another solution is possible. If $\overline{BC}$ is drawn, $\triangle ABC$ is an isosceles triangle and $m\angle ABC = 50$. Since $m\angle ABC = \frac{1}{2}m\overset{\frown}{BC}$, it follows that $m\overset{\frown}{BC} = 2m\angle ABC$. Hence, $m\overset{\frown}{BC} = 2(50) = 100$.

EXERCISES

1. Find the measure of the angle formed by two tangents drawn to a circle from an external point if they intercept a minor arc whose measure is:
a. 160 *b.* 140 *c.* 120 *d.* 145 *e.* 135 *f.* 90

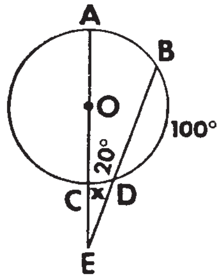
2. If two tangents drawn from an external point to a circle intercept a major arc whose measure is 210, the angle between the tangents measures ______.

3. One of the arcs intercepted by two tangents drawn to a circle from an external point measures 260. The measure of the angle formed by the tangents is ______.

4. If two tangents to a circle form an angle whose measure is 30, then the minor intercepted arc measures ______.

5. Tangents $\overline{PA}$ and $\overline{PB}$ from an external point P to circle O form an angle whose measure is 70. If radii $\overline{OA}$ and $\overline{OB}$ are drawn, what is the measure of angle AOB?
6. Two tangents $\overline{PA}$ and $\overline{PB}$ are drawn from point P to a circle. The measure of the major arc intercepted by the tangents is twice the measure of the minor arc. Find the measure of angle APB.
7. Find the measure of the angle formed by a tangent and a secant drawn to a circle from an external point if they intercept arcs whose measures are:
   a. 110 and 50 *b.* 120 and 65 *c.* a and b *d.* $6a$ and $2a$
8. The sides of an angle formed by a tangent and a secant drawn to a circle from an external point intercept arcs whose measures are 140 and 30. What is the measure of the angle?
9. An angle formed by a tangent and a secant drawn to a circle from an external point measures 65. If the greater intercepted arc measures 170, find the measure of the smaller intercepted arc.
10. An angle formed by a tangent and a secant drawn to a circle from an external point measures 40. Find the measure of the intercepted arcs if the measure of the larger arc is twice the measure of the smaller arc.
11. Find the measure of the angle formed by two secants drawn to a circle from an external point if they intercept arcs whose measures are:
   a. 100 and 40 *b.* 110 and 60 *c.* m and n *d.* $7x$ and $3x$
12. Two secants drawn to a circle from an external point P intercept arcs of 90° and 20° on the circle. Find the measure of angle P.
13. Two sides of an angle formed by two secants drawn to a circle from the same point intercept arcs of 144° and 92°. Find the measure of the angle.
14. From point P outside a circle, two secants $\overline{PAC}$ and $\overline{PBD}$ are drawn. Angle P contains 32° and minor arc $\overset{\frown}{AB}$ contains 50°. Find the number of degrees in minor arc $\overset{\frown}{CD}$.

In 15–19, O is the center of the circle. Tangents to the circle are marked "tan." Find the value of x, or x and y, as indicated.



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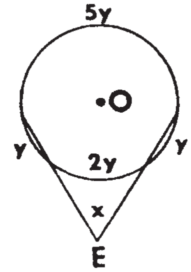

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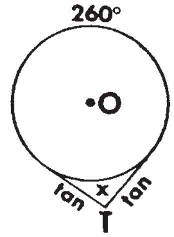

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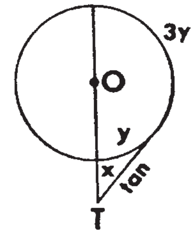

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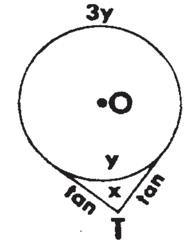
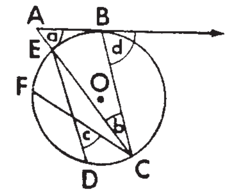


19.

More Difficult Exercises in Angle Measurements

MODEL PROBLEMS

1. In the figure, $\overleftrightarrow{AB}$ is a tangent to circle O at point B; $\overleftrightarrow{AEC}$ is a secant; and $\overline{DE}$, $\overline{FC}$, and $\overline{BC}$ are chords. $m\widehat{EB} = 50$, $m\widehat{BC} = (4x - 50)$, $m\widehat{CD} = x$, $m\widehat{DF} = (x + 25)$, $m\widehat{FE} = (x - 15)$.



a. Find $m\widehat{BC}$, $m\widehat{CD}$, $m\widehat{DF}$, and $m\widehat{FE}$.

b. Find $m\angle a$, $m\angle b$, $m\angle c$, $m\angle d$.

Solution:

a. 1. $m\widehat{EB} + m\widehat{BC} + m\widehat{CD} + m\widehat{DF} + m\widehat{FE} = 360$

2. $50 + 4x - 50 + x + x + 25 + x - 15 = 360$

3. $7x + 10 = 360$

4. $7x = 350$

5. $x = 50$

6. $m\widehat{BC} = 4x - 50 = 4(50) - 50 = 200 - 50 = 150$. *Ans.* $m\widehat{BC} = 150$.

7. $m\widehat{CD} = x = 50$. *Ans.* $m\widehat{CD} = 50$.

8. $m\widehat{DF} = x + 25 = 50 + 25 = 75$. *Ans.* $m\widehat{DF} = 75$.

9. $m\widehat{FE} = x - 15 = 50 - 15 = 35$. *Ans.* $m\widehat{FE} = 35$.

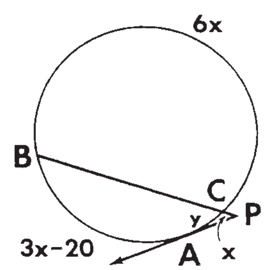
b. 10. $m\angle a = \frac{1}{2}(m\widehat{BC} - m\widehat{EB}) = \frac{1}{2}(150 - 50) = \frac{1}{2}(100) = 50$. *Ans.* $m\angle a = 50$.

11. $m\angle b = \frac{1}{2}m\widehat{EB} = \frac{1}{2}(50) = 25$. *Ans.* $m\angle b = 25$.

12. $m\angle c = \frac{1}{2}(m\widehat{CD} + m\widehat{FE}) = \frac{1}{2}(50 + 35) = \frac{1}{2}(85) = 42\frac{1}{2}$. *Ans.* $m\angle c = 42\frac{1}{2}$.

13. $m\angle d = \frac{1}{2}(m\widehat{BC}) = \frac{1}{2}(150) = 75$. *Ans.* $m\angle d = 75$.

2. $\overleftrightarrow{PA}$ is a tangent and $\overleftrightarrow{PCB}$ is a secant drawn to the circle. The measure of angle P is represented by x and the measures of $\widehat{AB}$, $\widehat{BC}$, and $\widehat{CA}$ are represented by $(3x - 20)$, $6x$, and y respectively.



a. In terms of x and y, write a set of equations that can be used to solve for x and y.

b. Solve the set of equations written in answer to part *a*.

c. Find the number of degrees contained in $\widehat{BAC}$.

[The solution is given on the next page.]

Solution:

a. 1. $m\angle BPA = \frac{1}{2}(m\widehat{BA} - m\widehat{AC})$

2. $x = \frac{1}{2}(3x - 20 - y)$

3. $2x = 3x - 20 - y$

4. $20 = 3x - 2x - y$

5. $20 = x - y$

6. $m\widehat{AC} + m\widehat{CB} + m\widehat{BA} = 360$

7. $y + 6x + 3x - 20 = 360$

8. $9x + y - 20 = 360$

9. $9x + y = 360 + 20$

10. $9x + y = 380$

Answer: A set of equations is $x - y = 20$
$9x + y = 380$

b. 11. $x - y = 20$

12. $9x + y = 380$ Add the corresponding members of equation 11 and equation 12.

13. $10x = 400$

14. $x = 40$

15. $9x + y = 380$

16. $9(40) + y = 380$ In equation 15, replace x by 40.

17. $360 + y = 380$

18. $y = 20$

Answer: $x = 40$, $y = 20$.

c. 19. $m\widehat{BAC} = m\widehat{BA} + m\widehat{AC}$

20. $m\widehat{BAC} = 3x - 20 + y$

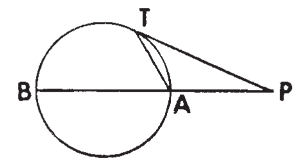
21. $m\widehat{BAC} = 3(40) - 20 + 20 = 120 - 20 + 20 = 120$

Answer: $\widehat{BAC}$ contains 120°.

EXERCISES

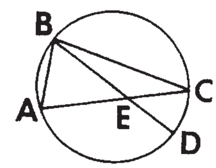
1. Quadrilateral $ABCD$ is inscribed in circle O; and the measures of arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, and $\overset{\frown}{DA}$ are in the ratio 2:5:3:8 respectively. (*a*) Find the measure of each arc. (*b*) Find $m\angle BAD$.

2. In the circle shown, $\overleftrightarrow{PT}$ is a tangent, $\overleftrightarrow{PAB}$ is a secant, and $\overline{TA}$ a chord. $m\overset{\frown}{BT} = 112$ and $m\angle PTA = 34$.
   a. Find the measure of angle BAT.
   b. Find the measure of angle TPB.
   c. Show that $\overline{AB}$ is a diameter of the circle.



Ex. 2

3. Chord $\overline{AC}$ and diameter $\overline{BD}$ intersect at E. $m\overset{\frown}{AB} = 56$ and $m\angle BAC = 72$. Find the measures of $\overset{\frown}{AD}$, $\overset{\frown}{BC}$, angle BEC, and angle ABC.

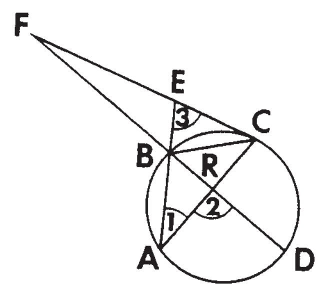


Ex. 3

4. The tangent to the circle at C intersects chord $\overline{DB}$ extended at F; chord $\overline{AB}$ extended intersects $\overline{FC}$ at E; $\overline{DB}$ intersects $\overline{AC}$ at R.

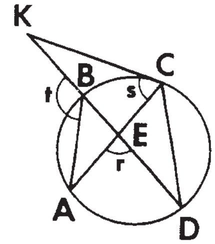
$$m\overset{\frown}{AB}:m\overset{\frown}{BC}:m\overset{\frown}{CD}:m\overset{\frown}{DA} = 8:7:10:11$$

   a. Find the measures of $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, and $\overset{\frown}{DA}$.
   b. Find $m\angle 1$, $m\angle 2$, and $m\angle 3$.



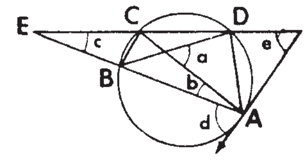
Ex. 4

5. $\overline{AB}$ and $\overline{CD}$ are chords. Chords $\overline{AC}$ and $\overline{DB}$ intersect at E. The tangent at C intersects $\overline{DB}$ extended at K. $m\overset{\frown}{AB} = 80$; and the measures of arcs $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, and $\overset{\frown}{DA}$ are represented by x, $(2x - 8)$, and $(x + 32)$ respectively.
   a. Find $m\overset{\frown}{BC}$, $m\overset{\frown}{CD}$, and $m\overset{\frown}{DA}$.
   b. Find $m\angle r$, $m\angle s$, and $m\angle t$.



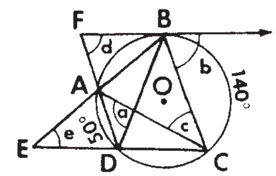
Ex. 5

6. $ABCD$ is a quadrilateral inscribed in a circle; $m\widehat{AB} = (4x - 25)$, $m\widehat{BC} = x$, $m\widehat{CD} = (2x + 20)$, $m\widehat{DA} = (3x - 35)$. Chords $\overline{AC}$ and $\overline{BD}$ are drawn; also chords $\overline{AB}$ and $\overline{DC}$ are extended to intersect at E, and the tangent at A intersects $\overline{CD}$ extended at F. Find the number of degrees contained in angle a, angle b, angle c, angle d, and angle e.



Ex. 6

7. $ABCD$ is a quadrilateral inscribed in circle O. Chord $\overline{BA} \cong$ chord $\overline{CD}$, and $\overline{BA}$ and $\overline{CD}$ extended intersect in point E. A tangent at B intersects $\overline{DA}$ extended in point F. Diagonals $\overline{BD}$ and $\overline{AC}$ are drawn. $m\widehat{AD} = 50$ and $m\widehat{BC} = 140$. Find $m\angle a$, $m\angle b$, $m\angle c$, $m\angle d$, and $m\angle e$.



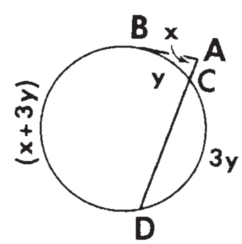
Ex. 7

8. Two secants $\overline{PDA}$ and $\overline{PCB}$ are drawn from external point P to a circle. Chords $\overline{AC}$ and $\overline{BD}$ intersect in F. If $m\widehat{AB} = 128$ and $m\widehat{CD} = 32$, find $m\angle P$, $m\angle ADB$, and $m\angle BFC$.

9. Tangent $\overleftrightarrow{PC}$ and secant $\overleftrightarrow{PBA}$ are drawn from point P to a circle, and chords $\overline{BC}$ and $\overline{AC}$ are drawn. $m\angle APC = 37$ and $m\widehat{ABC} = 236$. Find $m\widehat{BC}$, $m\angle BCP$, and $m\angle ACB$.

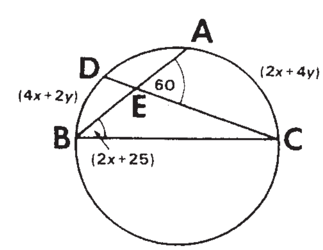
10. Congruent chords $\overline{AB}$ and $\overline{CD}$ of a circle are extended through B and D to intersect in P. $\overline{AD}$ and $\overline{BC}$ intersect in E, and $\overline{AC}$ is drawn. $m\angle P = 18$ and $m\widehat{AC} = 60$. Find $m\widehat{BD}$, $m\angle BCD$, $m\angle AEC$, and $m\angle ACB$.

11. $\overleftrightarrow{AB}$ is a tangent and $\overleftrightarrow{ACD}$ is a secant to a circle. $m\angle A$ is represented by x and $m\widehat{DB}$, $m\widehat{BC}$, and $m\widehat{CD}$ are represented by $(x + 3y)$, y, and $3y$ respectively.
   a. In terms of x and y, write a set of equations that can be used to solve for x and y.
   b. Solve the set of equations written in part *a*.
   c. Find the number of degrees contained in arc $\widehat{DBC}$.



Ex. 11

12. Chords $\overline{AB}$ and $\overline{CD}$ intersect at point E, and chord $\overline{BC}$ is drawn. The measures of arcs $\widehat{AC}$ and $\widehat{BD}$ are represented by $(2x + 4y)$ and $(4x + 2y)$ respectively. The measure of angle ABC is represented by $(2x + 25)$ and the measure of angle AEC is 60.
   a. In terms of x and y, write a set of equations that can be used to solve for x and y.



Ex. 12

b. Solve the set of equations written in answer to *a* to obtain the values of x and y.

c. Find $m\widehat{AC}$ and $m\widehat{BD}$.

13. Secants $\overline{PQR}$ and $\overline{PST}$ are drawn to a circle from external point P so that $m\angle P = 20$ and $m\angle QTS = 10$. If $m\widehat{QS}$ is represented by $(3x + y)$ and $m\widehat{RT}$ by $(8x + 4y)$:

a. Write a pair of equations that can be used to solve for x and y.

b. Solve these equations to find values for x and y.

c. Find $m\angle RQT$.

14. The sides of a triangle inscribed in a circle have arcs whose measures are represented by $(x + 15)$, $(6x + 10)$, and $(8x - 40)$. Show that the triangle is an isosceles triangle.

15. The sides of a quadrilateral inscribed in a circle have arcs whose measures are represented by $(4x - 6)$, $(2x + 42)$, $(5x - 30)$, and $(3x + 18)$. Show that the quadrilateral is a square.

6. Using Angle Measurement in Proving Angles Congruent and Arcs Equal

Summary of Methods of Proving Angles Congruent and Arcs Equal

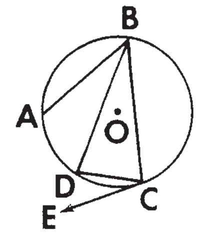
To prove that two or more angles are congruent, prove that any one of the following statements is true:

1. The measures of the angles are equal.
2. The angles are central angles that intercept the same arc or equal arcs in a circle or in equal circles.
3. The angles are inscribed in the same arc or in equal arcs in a circle or in equal circles.
4. The angles are inscribed angles or angles formed by a tangent and a chord that intercept the same arc or equal arcs in a circle or in equal circles.

To prove that two or more arcs are equal, prove that any one of the following statements is true:

1. The measures of the arcs are equal.
2. The arcs are intercepted by congruent central angles in a circle or in equal circles.
3. The arcs are intercepted by congruent inscribed angles, or congruent angles formed by a tangent and a chord, in a circle or in equal circles.

MODEL PROBLEMS



1. *Given:* In circle O, $\overline{BD}$ bisects inscribed $\angle ABC$. $\overleftrightarrow{EC}$ is a tangent to circle O at C.

To prove: $\angle ABD \cong \angle DCE$.

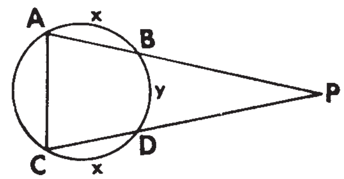
Plan: To prove $\angle ABD \cong \angle DCE$, first use the "given" to prove that $\angle ABD \cong \angle DBC$. Then prove that $\angle DBC \cong \angle DCE$ by showing that $m\angle DBC = m\angle DCE$, since the measure of each angle is equal to $\frac{1}{2}m\overset{\frown}{DC}$. Therefore, $\angle ABD \cong \angle DCE$ by the transitive property of congruence.

Proof:

Statements	*Reasons*
1. $\overline{BD}$ bisects $\angle ABC$.	1. Given.
2. $\angle ABD \cong \angle DBC$.	2. A bisector divides an angle into two congruent angles.
3. $\overleftrightarrow{EC}$ is tangent to circle O.	3. Given.
4. $m\angle DCE = \frac{1}{2}m\overset{\frown}{DC}$.	4. The measure of an angle formed by a tangent and a chord at the point of contact is equal to one-half the measure of the intercepted arc.
5. $m\angle DBC = \frac{1}{2}m\overset{\frown}{DC}$.	5. The measure of an inscribed angle is equal to one-half the measure of the intercepted arc.
6. $m\angle DBC = m\angle DCE$.	6. Transitive property of equality.
7. $\angle DBC \cong \angle DCE$.	7. Two angles are congruent if their measures are equal.
8. $\angle ABD \cong \angle DCE$.	8. Transitive property of congruence.

2. In a circle, congruent chords $\overline{AB}$ and $\overline{CD}$ are extended through B and D respectively until they intersect at P. Prove that triangle APC is an isosceles triangle.

Given: Chord $\overline{AB} \cong$ chord $\overline{CD}$. Chords $\overline{AB}$ and $\overline{CD}$ are extended to intersect at P.



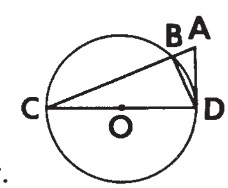
To prove: $\triangle APC$ is an isosceles triangle.

Plan: To prove that $\triangle APC$ is an isosceles triangle, show that $\angle A \cong \angle C$. It can be shown that $\angle A \cong \angle C$ by proving that both angles are inscribed angles whose measures are equal to one-half the measures of the equal arcs $\overset{\frown}{CDB}$ and $\overset{\frown}{ABD}$. Let $m\overset{\frown}{AB} = x$ and $m\overset{\frown}{BD} = y$.

Proof:

Statements	*Reasons*
1. Chord $\overline{AB} \cong$ chord $\overline{CD}$.	1. Given.
2. $m\overset{\frown}{AB} = m\overset{\frown}{CD} = x$.	2. In a circle, congruent chords have equal arcs.
3. $m\overset{\frown}{ABD} = x + y$, and $m\overset{\frown}{CDB} = x + y$.	3. The measure of an arc is equal to the sum of the measures of its parts.
4. $m\overset{\frown}{ABD} = m\overset{\frown}{CDB}$.	4. Transitive property of equality.
5. $\angle A \cong \angle C$.	5. In a circle, if inscribed angles intercept equal arcs, the angles are congruent.
6. $\overline{PA} \cong \overline{PC}$.	6. In a triangle, if two angles are congruent, the sides opposite these angles are congruent.
7. $\triangle APC$ is an isosceles triangle.	7. An isosceles triangle is a triangle that has two congruent sides.

3. *Given:* $\overline{CD}$ is a diameter in circle O.
$\overleftrightarrow{AD}$ is tangent to circle O.
$\overleftrightarrow{ABC}$ is a line.



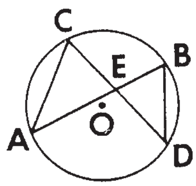
To prove: $\triangle ABD$ and $\triangle DBC$ are mutually equiangular.

Plan: To prove that $\triangle ABD$ and $\triangle DBC$ are mutually equiangular means that we must prove that three angles in $\triangle ABD$ are congruent to three angles in $\triangle DBC$.

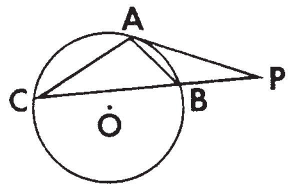
[The proof is given on the next page.]

Proof: *Statements*	*Reasons*
1. $\overline{CD}$ is a diameter in circle O.	1. Given.
2. $\angle CBD$ is a right angle.	2. An angle inscribed in a semicircle is a right angle.
3. $\overleftrightarrow{ABC}$ is a line.	3. Given.
4. $\angle ABD$ is supplementary to $\angle CBD$.	4. If two adjacent angles have their non-common sides on a straight line, the angles are supplementary.
5. $\angle ABD$ is a right angle, or $m\angle ABD = 90$.	5. The supplement of a right angle is a right angle.
6. $\angle CBD \cong \angle ABD$. (a. $\cong$ a.)	6. All right angles are congruent.
7. $\overleftrightarrow{AD}$ is tangent to circle O.	7. Given.
8. $\angle CDA$ is a right angle.	8. A tangent to a circle is perpendicular to a radius at the point of contact.
9. $\angle ADB$ is complementary to $\angle BDC$.	9. Two angles are complementary if the sum of their measures is 90.
10. $\angle BCD$ is complementary to $\angle BDC$.	10. The acute angles of a right triangle are complementary.
11. $\angle BCD \cong \angle ADB$. (a. $\cong$ a.)	11. If two angles are complements of the same angle, they are congruent.
12. $\angle BDC \cong \angle BAD$. (a. $\cong$ a.)	12. If two angles in one triangle are congruent to two angles in another triangle, the third angles in these triangles are congruent.
13. $\triangle ABD$ and $\triangle DBC$ are mutually equiangular.	13. If three angles in one triangle are congruent to three angles in another triangle, the triangles are mutually equiangular.

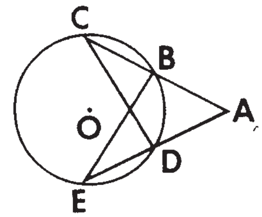
EXERCISES



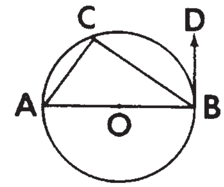
Ex. 1



Ex. 2

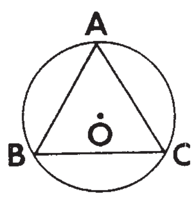


Ex. 3

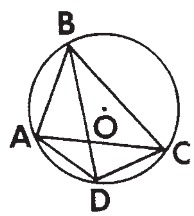


Ex. 4

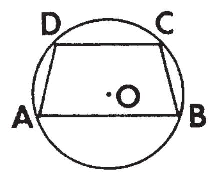
1. In circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect in E. Prove that $\angle CAB \cong \angle BDC$.
2. If $\overline{PBC}$ is a secant and $\overleftrightarrow{PA}$ is a tangent to circle O at A, prove that $\angle PAB \cong \angle ACB$.
3. In circle O, $\overline{ABC}$ and $\overline{ADE}$ are secants. Prove that $\angle ADC \cong \angle ABE$.
4. In circle O, $\overline{AB}$ is a diameter. $\overleftrightarrow{DB}$ is tangent to circle O at B. Prove that $\angle ACB \cong \angle ABD$.



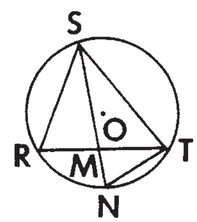
Ex. 5



Ex. 6



Ex. 7



Ex. 8

5. Triangle ABC is inscribed in circle O. If $\angle B \cong \angle C$, prove that $\widehat{ABC} = \widehat{ACB}$.
6. In circle O, $\overline{BD}$ bisects inscribed $\angle ABC$. Prove that $\triangle ADC$ is isosceles.
7. Quadrilateral $ABCD$ is inscribed in circle O, and $\angle DAB \cong \angle CBA$. (*a*) Prove: $\overline{DA} \cong \overline{CB}$. (*b*) Prove: $\overline{DC} \parallel \overline{AB}$.
8. Triangle RST is inscribed in circle O. $\overline{SN}$ bisects $\angle RST$. Prove: $\angle SMR \cong \angle STN$.
9. *Prove:* The bisector of an angle inscribed in a circle bisects the arc intercepted by the angle.
10. *Prove:* A trapezoid inscribed in a circle is isosceles.
11. Isosceles triangle ABC with $\overline{AB} \cong \overline{AC}$ is inscribed in a circle. Prove that if $\overleftrightarrow{ED}$ is a tangent to the circle at vertex A, $\overleftrightarrow{ED}$ is parallel to $\overleftrightarrow{BC}$.
12. *Prove:* A parallelogram inscribed in a circle is a rectangle.

13. In a circle, chord $\overline{AB}$ is parallel to tangent $\overleftrightarrow{CD}$, which intersects the circle at P. Prove that angle CPA is congruent to angle DPB.

14. From P, a point outside a circle, congruent secants $\overline{PAB}$ and $\overline{PCD}$ are drawn to the circle. Prove that chord $\overline{AB} \cong$ chord $\overline{CD}$.

15. In circle O, $\overline{AB}$ is a diameter and $\overline{AC}$ is a chord. Prove that $\overline{OD}$, the radius which bisects arc $\overset{\frown}{CB}$, is parallel to chord $\overline{AC}$.

16. A, B, C, and D are four points taken consecutively on a circle and so located that the measure of $\overset{\frown}{BC}$ is twice each of the measures of $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$. Chords $\overline{AC}$ and $\overline{BD}$ are drawn intersecting in M, and chord $\overline{DC}$ is drawn. Prove that triangle DCM is isosceles.

17. $\overline{AB}$ is a diameter of a circle and C is any point on the circle. Chord $\overline{AC}$ is drawn and extended to D so that $\overline{AC} \cong \overline{CD}$, and $\overline{DB}$ is drawn. Prove that $\overline{DB} \cong \overline{AB}$.

18. Angle ABC is inscribed in a circle. Chord $\overline{BD}$ bisects angle ABC, and chord $\overline{DE}$ is drawn parallel to $\overline{AB}$. Prove that chord $\overline{DE}$ is congruent to chord $\overline{BC}$.

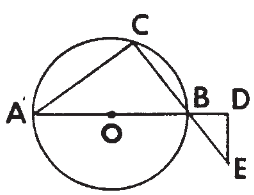
19. Quadrilateral $ABCD$ is inscribed in a circle. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect in E, and $\overline{BE} \cong \overline{CE}$. Prove that chord $\overline{AB} \cong$ chord $\overline{CD}$.

20. *Prove:* A parallelogram circumscribed about a circle is a rhombus.

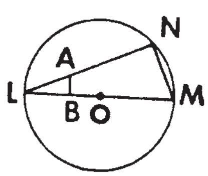
21. In a circle, chords $\overline{LM}$ and $\overline{RS}$ intersect at P. Prove that three angles in $\triangle LPR$ are congruent to three angles in $\triangle SPM$.

22. From point P outside a circle, tangent $\overleftrightarrow{PA}$, A being the point of tangency, and secant $\overline{PBC}$ are drawn. Prove that three angles in $\triangle PAB$ are congruent to three angles in $\triangle PCA$.

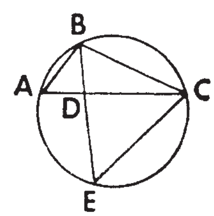
23. From point P outside a circle, two secants $\overline{PAB}$ and $\overline{PCD}$ are drawn. Prove that $\triangle PAD$ and $\triangle PCB$ are mutually equiangular.



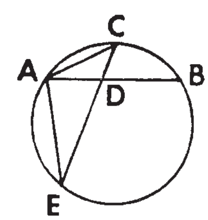
Ex. 24



Ex. 25



Ex. 26



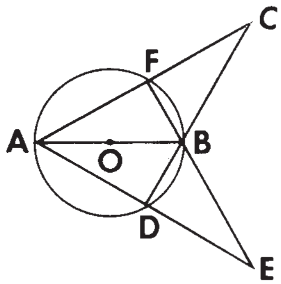
Ex. 27

24. In circle O, diameter $\overline{AB}$ is extended through B to D, $\overline{DE} \perp \overline{AD}$, and $\overleftrightarrow{CBE}$ is a line. Prove that $\triangle ACB$ and $\triangle EDB$ are mutually equiangular.

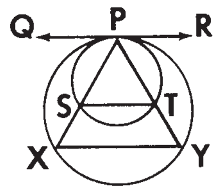
25. In circle O, $\overline{LM}$ is a diameter and $\overline{AB} \perp \overline{LM}$. Prove that three angles in $\triangle ABL$ are congruent to three angles in $\triangle MNL$.

26. Triangle ABC is inscribed in a circle. $\overline{BDE}$ is the bisector of angle B. Prove that $\triangle ABD$ and $\triangle EBC$ are mutually equiangular.

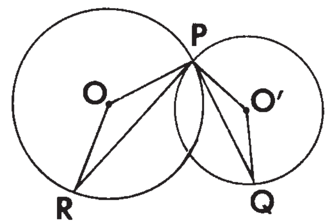
27. C is the midpoint of $\overset{\frown}{AB}$. $\overleftrightarrow{CDE}$ is a line. Prove that three angles in $\triangle ACD$ are congruent to three angles in $\triangle ECA$.



Ex. 28



Ex. 29



Ex. 30

28. If $\overline{AB}$ is a diameter in circle O and $\overline{CA}$, $\overline{CD}$, $\overline{EF}$, and $\overline{EA}$ are secants, prove that $\triangle DBE$ and $\triangle FBC$ are mutually equiangular.

29. $\overleftrightarrow{RQ}$ is the common tangent to two circles that are internally tangent at P. $\overline{PX}$ and $\overline{PY}$ are chords of the larger circle which intersect the smaller circle in S and T respectively. (*a*) Prove that $\overline{ST} \parallel \overline{XY}$. (*b*) Prove that $\triangle PST$ and $\triangle PXY$ are mutually equiangular.

30. Unequal circles O and O' intersect at P. Chord $\overline{PR}$ is tangent to circle O' and chord $\overline{PQ}$ is tangent to circle O. Radii $\overline{OP}$, $\overline{OR}$, $\overline{O'P}$, and $\overline{O'Q}$ are drawn. (*a*) Prove that $\angle OPR \cong \angle O'PQ$. (*b*) Prove that $\triangle OPR$ and $\triangle O'PQ$ are mutually equiangular.

7. Completion Exercises

Write a word or expression that, when inserted in the blank, will make the resulting statement true.

1. The measure of an angle formed by a tangent and a secant intersecting in the exterior region of a circle is equal to one-half the ______ of the intercepted arcs.

2. The measure of an angle formed by two chords intersecting in the interior region of a circle is equal to one-half the ______ of the intercepted arcs.

3. The measure of an angle formed by two secants intersecting in the exterior region of a circle is equal to one-half the ______ of the intercepted arcs.

4. The measure of an angle formed by a tangent and a chord drawn to the point of contact is equal to ______ its intercepted arc.

5. If two chords of a circle bisect each other, the chords must be ______ of the circle.

6. If quadrilateral $ABCD$ is inscribed in a circle O, the sum of the measures of angles A and C is ______ degrees.

7. In a circle, a central angle and an inscribed angle intercept the same arc. The ratio of the measure of the central angle to the measure of the inscribed angle is ______.

8. An angle inscribed in a(an) ______ is a right angle.
9. An angle inscribed in an arc which contains less than 180° is a(an) ______ angle.
10. An angle inscribed in an arc whose measure is greater than the measure of a semicircle is a(an) ______ angle.
11. In a circle, if two inscribed angles intercept the same arc, the ratio of their measures is ______.
12. If two circles are tangent externally, the distance between their centers is equal to the ______ of the lengths of their radii.
13. Tangents to a circle at the ends of a diameter are ______ to each other.
14. In a circle, a chord which is the perpendicular bisector of another chord must be a(an) ______.
15. Two externally tangent circles have ______ common tangents.
16. In two concentric circles, all chords of the larger circle which are tangent to the smaller circle are ______.
17. Chords $\overline{AB}$ and $\overline{CD}$ of a circle intersect in E. If $m\widehat{AD} = 6x$ and $m\widehat{BC} = 4x$, then angle AED contains ______ degrees.
18. To circumscribe a circle about a triangle, it is necessary to bisect two of the ______ of the triangle.
19. The center of a circle circumscribed about a triangle is equidistant from the three ______ of the triangle.
20. To inscribe a circle in a triangle, it is necessary to bisect two of the ______ of the triangle.
21. The hypotenuse of a right triangle which is inscribed in a circle is ______ times as long as the radius of the circle.
22. If a parallelogram is inscribed in a circle, then the parallelogram must be a(an) ______.
23. If a line is perpendicular to a radius of a circle at its outer extremity, then the line is ______ to the circle.
24. Two internally tangent circles have ______ common tangent(s).
25. Two parallel chords intercept ______ arc(s) on a circle.

8. True-False Exercises

If the statement is always true, write *true;* if the statement is not always true, write *false*.

1. If chord $\overline{LM}$ bisects chord $\overline{AB}$, then $\overline{LM}$ must be perpendicular to $\overline{AB}$.

2. The length of the median to the hypotenuse of a right triangle is equal to half the length of the hypotenuse.
3. A bisector of a chord passes through the center of the circle.
4. The bisector of an inscribed angle bisects the intercepted arc.
5. In equal circles, congruent central angles intercept arcs of the same number of degrees.
6. If in a circle point P is the midpoint of $\overset{\frown}{CD}$, then the length of chord $\overline{CP}$ is one-half the length of chord $\overline{CD}$.
7. If a secant and a tangent to a circle are parallel, the diameter drawn to the point of tangency is perpendicular to the secant.
8. A parallelogram inscribed in a circle is a rectangle.
9. A trapezoid inscribed in a circle is isosceles.
10. A diameter which bisects one of two parallel chords, neither of which is a diameter, bisects the other chord also.
11. If parallelogram $ABCD$ is inscribed in a circle, sides $\overline{AB}$ and $\overline{DC}$ are equidistant from the center of the circle.
12. If two chords intercept equal arcs on a circle, the chords are parallel.
13. An angle inscribed in an arc which contains more than 180° is an obtuse angle.
14. A line which passes through the midpoint of a chord and the midpoint of its minor arc passes through the center of the circle.
15. If two circles are concentric, chords of the larger circle which are tangent to the smaller circle are congruent.
16. The opposite angles of an inscribed quadrilateral are complementary.
17. If two chords intersecting in a circle intercept opposite arcs the sum of whose measures is 180, then the chords are perpendicular to each other.
18. If two chords of a circle are perpendicular to each other, then one chord is a diameter.
19. There are two and only two circles of radius r which are tangent to each of two intersecting lines.
20. Congruent chords must have equal arcs.
21. If two circles are externally tangent to each other, the greatest number of common tangents that can be drawn to both circles is four.
22. If two chords intercept equal arcs on a circle, the chords are equidistant from the center of the circle.
23. If a circle is circumscribed about a parallelogram, each diagonal of the parallelogram is a diameter of the circle.
24. An angle inscribed in a major arc of a circle is an obtuse angle.
25. If two circles do not intersect, they must have four common tangents.

9. "Always, Sometimes, Never" Exercises

If the blank space in each of the following exercises is replaced by the word *always*, *sometimes*, or *never*, the resulting statement will be true. Select the word which will correctly complete each statement.

1. If two arcs of a circle are equal, the chords of these arcs are _______ parallel.
2. If the perpendiculars drawn from the center of a circle upon two chords are congruent, then the minor arcs subtended by these chords are _______ equal.
3. The perpendicular bisector of a chord of a circle _______ passes through the center of the circle.
4. A line passing through the midpoint of a chord of a circle _______ passes through the center of the circle.
5. If in a given circle $\widehat{AB}$ equals $\widehat{BC}$, then the length of chord $\overline{AC}$ is _______ twice the length of chord $\overline{AB}$.
6. Tangents drawn from an external point to a circle _______ make congruent angles with the chord joining the points of tangency.
7. Chord $\overline{AB}$ of circle O passes through the midpoints of two parallel chords. $\overline{AB}$ is _______ a diameter of the circle.
8. If in the same circle or in equal circles two chords are congruent, they are _______ equidistant from the center.
9. A parallelogram inscribed in a circle _______ has two acute angles.
10. If two chords of a circle are perpendicular to a third chord at its endpoints, the two chords are _______ congruent.
11. If the perpendicular bisectors of the sides of a triangle intersect on a side of the triangle, the triangle is _______ a right triangle.
12. Quadrilateral $ABCD$ is circumscribed about a circle whose center is O. The bisector of angle A _______ passes through O.
13. If from point A on a circle chord $\overline{AB}$ and tangent $\overleftrightarrow{AP}$ are drawn, then $m\angle PAB$ is _______ less than 90.
14. If two circles are concentric, any two chords of the larger circle which are tangent to the smaller circle are _______ congruent.
15. From external point A, tangents $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are drawn to a circle, and chord $\overline{BC}$ is drawn. Triangle ABC is _______ equilateral.
16. If two chords of a circle are parallel, the minor arc of one chord is _______ equal to the minor arc of the other.
17. If the measure of the angle between two tangents to a circle is 60, the

triangle formed by these tangents and the chord joining the points of contact is _______ equiangular.

18. Two externally tangent circles _______ have three common tangents.
19. Two right triangles inscribed in the same or equal circles are _______ congruent.
20. Two triangles are inscribed in congruent circles and have their angles respectively congruent. The triangles are _______ congruent.
21. In circle O, chord $\overline{AB}$ is drawn. The length of $\widehat{AB}$ is _______ equal to the length of chord $\overline{AB}$.
22. An inscribed angle which intercepts an arc whose measure is less than 180 is _______ an obtuse angle.
23. An angle inscribed in an arc whose measure is less than 180 is _______ an obtuse angle.
24. If two angles intercept the same arc of a circle, they are _______ congruent.
25. If a chord in one circle is congruent to a chord in another circle, the minor arcs of these chords are _______ congruent.
26. At least one common tangent can _______ be drawn to two given circles.
27. A circle can _______ be constructed which will pass through three given points.
28. The center of the circle circumscribed about a triangle is _______ in the exterior region of the triangle.
29. The center of the circle circumscribed about a triangle _______ lies in the interior region of the triangle.
30. The bisectors of the angles of a triangle are collinear at a point which is _______ equidistant from the three sides of the triangle.

10. Multiple-Choice Exercises

Write the letter preceding the word or expression that best completes the statement.

1. In a circle, a central angle of $x°$ intercepts an arc of (*a*) $\frac{1}{2}x°$ (*b*) $x°$ (*c*) $2x°$.
2. If an angle inscribed in a circle intercepts an arc of $y°$, the angle contains (*a*) $\frac{1}{2}y°$ (*b*) $y°$ (*c*) $2y°$.
3. If two tangents are drawn to a circle at the ends of a chord whose arc measures 100°, the triangle formed is (*a*) acute (*b*) right (*c*) obtuse.
4. The angle whose measure is equal to one-half the difference of the meas-

ures of its intercepted arcs has its vertex (*a*) in the interior region of the circle (*b*) on the circle (*c*) in the exterior region of the circle.

5. The angle whose measure is equal to one-half the sum of the measures of its intercepted arcs has its vertex (*a*) in the exterior region of the circle (*b*) in the interior region of the circle (*c*) on the circle.
6. An angle inscribed in an arc which measures less than 180° is (*a*) acute (*b*) right (*c*) obtuse.
7. The opposite angles of a quadrilateral inscribed in a circle are always (*a*) congruent (*b*) complementary (*c*) supplementary.
8. The two chords that form the sides of an angle inscribed in a semicircle are always (*a*) congruent (*b*) unequal in length (*c*) perpendicular to each other.
9. A circle can always be circumscribed about (*a*) a parallelogram (*b*) a rectangle (*c*) a rhombus.
10. If the length of a median of a triangle is equal to one-half the length of the side to which it is drawn, the triangle is (*a*) acute (*b*) obtuse (*c*) right.
11. If a parallelogram is inscribed in a circle, the diagonals always (*a*) are not congruent (*b*) are diameters (*c*) are perpendicular to each other.
12. If two chords intersecting within a circle intercept opposite arcs the sum of whose measures is 180, the angle formed by the two chords is (*a*) acute (*b*) right (*c*) obtuse.
13. If two circles have a common chord, then the sum of the lengths of their radii is (*a*) greater than (*b*) equal to (*c*) less than the distance between their centers.
14. If for two given circles only two common tangents exist, the circles (*a*) intersect in two points (*b*) are tangent internally (*c*) are tangent externally.
15. If two circles are tangent internally, the distance between their centers is equal to (*a*) the sum of the lengths of their radii (*b*) the difference of the lengths of their radii (*c*) the product of the lengths of their radii.
16. Two circles are externally tangent. The number of common tangents which these circles can have is (*a*) one (*b*) three (*c*) four.
17. The number of circles that can be tangent to each of two intersecting lines is (*a*) two (*b*) four (*c*) infinite.
18. Two unequal circles are tangent externally. From a point on their common internal tangent, tangents are drawn to the circles. The length of the tangent to the larger circle is (*a*) greater than the length of the tangent to the smaller circle (*b*) equal to the length of the tangent to the smaller circle (*c*) less than the length of the tangent to the smaller circle.

19. If two tangents are drawn to a circle at the endpoints of a chord which has an arc of 60°, the triangle formed by the tangents and the chord is (*a*) acute (*b*) equilateral (*c*) obtuse.

20. If A, B, C, and D are four consecutive points on a circle such that $\widehat{AB} = \widehat{CD}$, then chords $\overline{BC}$ and $\overline{AD}$ always (*a*) are congruent (*b*) intersect (*c*) are parallel.

21. In a circle, an inscribed angle and a central angle intercept the same arc. The ratio of the number of degrees in the inscribed angle to the number of degrees in the central angle is (*a*) 1:2 (*b*) 1:1 (*c*) 2:1.

22. Two circles with radii 5 and 10 respectively are internally tangent to each other. The distance between their centers is (*a*) 2 (*b*) 15 (*c*) 5.

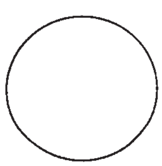
23. Tangents $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are drawn from external point P to circle O, points A and B being the points of contact. Angle PAB is always (*a*) an acute angle (*b*) the supplement of angle AOB (*c*) congruent to angle APB.

24. The center of a circle which circumscribes a triangle is always the point of concurrency of (*a*) the altitudes of the triangle (*b*) the bisectors of the angles of the triangle (*c*) the perpendicular bisectors of the sides of the triangle.

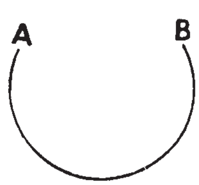
25. The center of a circle circumscribed about a triangle lies in the interior region of the triangle if the triangle is (*a*) acute (*b*) right (*c*) obtuse.

11. Construction Exercises

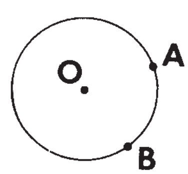
The basic constructions involved in the following exercises, which are to be done with straightedge and compasses, appear in Chapter 13, which begins on page 614.



Ex. 1



Ex. 2



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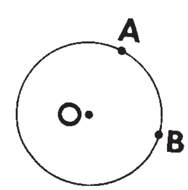
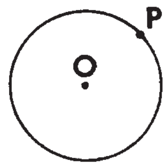
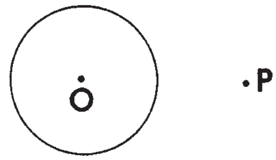


Ex. 4

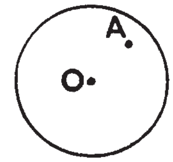
1. Find by construction the center of the circle.
2. Find by construction the center of the circle of which $\widehat{AB}$ is a part.
3. Locate by construction the midpoint M of minor $\widehat{AB}$ in circle O.
4. Locate by construction two points M and N on circle O each of which is equidistant from points A and B.



Ex. 5

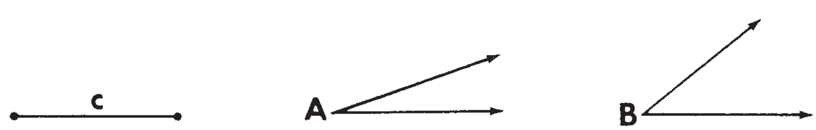


Ex. 6



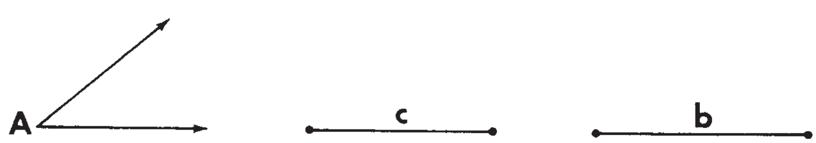
Ex. 7

5. Construct a tangent to circle O at point P.
6. Through point P; construct a tangent to circle O.
7. Through point A, construct the chord of circle O whose midpoint is A.
8. Through a point inside a circle, construct the smallest chord.
9. In a given circle, construct two chords that are congruent and parallel.
10. Construct two parallel lines which are tangent to the same circle.
11. Construct a tangent to a circle which is parallel to a given line in the exterior region of the circle.
12. Find the center of the circle that can be inscribed in a given (*a*) acute triangle (*b*) obtuse triangle (*c*) right triangle.
13. Find the center of the circle that can be circumscribed about a given (*a*) acute triangle (*b*) obtuse triangle (*c*) right triangle.
14. Inscribe a circle in a given (*a*) acute triangle (*b*) obtuse triangle (*c*) right triangle.
15. Circumscribe a circle about a given (*a*) acute triangle (*b*) obtuse triangle (*c*) right triangle.
16. Circumscribe a circle about a given rectangle.
17. Inscribe a circle in a given rhombus.
18. Inscribe an equilateral triangle in a given circle.
19. Construct a circle which will pass through three given points A, B, and C which are not collinear.
20. Construct triangle ABC, having given angle A, angle B, and the included side c.



Ex. 20

21. Construct triangle RST in which $\angle R \cong \angle A$, $RS = c$, and $RT = b$.

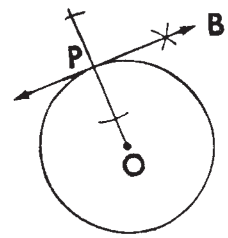


Ex. 21

22. Construct an isosceles triangle whose base is the given line segment b and whose altitude upon b is the given line segment h.

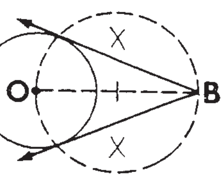
Ex. 22

23. To construct a tangent to circle O at point P, a line is drawn perpendicular to $\overleftrightarrow{OP}$ at point P as shown in the diagram. Which one of the following statements is the theorem used to prove that $\overleftrightarrow{BP}$ is tangent to circle O?
   a. A tangent to a circle is a line in the plane of the circle which intersects the circle in one and only one point.
   b. A line perpendicular to a radius at its endpoint on the circle is tangent to the circle.
   c. A tangent to a circle is perpendicular to the radius drawn to the point of contact.



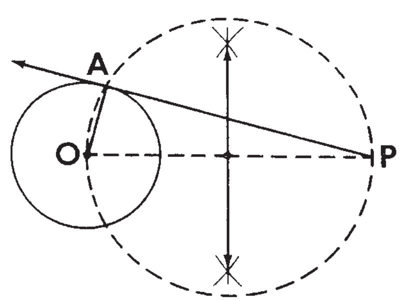
Ex. 23

24. The construction of tangents to circle O from external point B is shown. Which one of the following statements is used to prove the construction?
   a. In a circle, a line perpendicular to a radius at its outer extremity is tangent to the circle.
   b. A tangent to a circle is perpendicular to the radius drawn to the point of contact.
   c. A tangent to a circle is a line in the plane of the circle which intersects the circle in one and only one point.



Ex. 24

25. The diagram shows the construction of a tangent $\overleftrightarrow{PA}$ to circle O from external point P. Which theorem is used in the proof of this construction to show that $\overleftrightarrow{PA}$ is a tangent?
   a. If two circles intersect, the line joining their centers is the perpendicular bisector of the common chord.
   b. An angle inscribed in a semicircle is a right angle.
   c. Tangents to a circle from an external point are congruent.



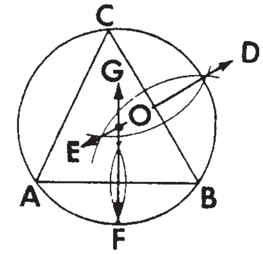
Ex. 25

26. The construction for circumscribing a circle about a given triangle is shown. Which two of the following statements are used in proving the construction?

a. If two quantities are equal to the same quantity, they are equal to each other.

b. If a point is equidistant from the endpoints of a line segment, then the point is on the perpendicular bisector of the line segment.

c. If a point is on the perpendicular bisector of a line segment, then the point is equidistant from the endpoints of the line segment.



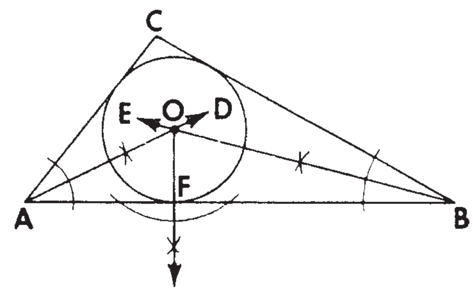
Ex. 26

27. The construction for inscribing a circle in a given triangle is shown. Which two of the following statements are used in proving the construction?

a. If a point is on the bisector of an angle, the point is equidistant from the sides of the angle.

b. If two quantities are equal to the same quantity, they are equal to each other.

c. If a point is equidistant from the sides of an angle, then the point is on the bisector of the angle.



Ex. 27