CHAPTER IX

Inequalities

1. Symbols of Inequality

Up to this point, we have devoted a great deal of our attention to proving line segments congruent, angles congruent, measures of lines equal, and measures of angles equal. Sometimes we have talked about line segments or angles whose measures are unequal. Now we will study (1) methods of proving that the measure of one line segment is greater than the measure of another line segment, and (2) methods of proving that the measure of one angle is greater than the measure of another angle; that is, we will concern ourselves with the comparison of sizes of line segments and sizes of angles.

Let us recall the meanings of some of the symbols that you probably used in the study of algebra to indicate the inequality of two real numbers.

$4 \neq 3$ means 4 is not equal to 3	$a \neq b$ means a is not equal to b
$8 > 6$ means 8 is greater than 6	$a > b$ means a is greater than b
$5 < 7$ means 5 is less than 7	$a < b$ means a is less than b

A statement of inequality may be a true statement or it may be a false statement. For example, $5 \neq 4$, $8 > 7$, and $6 < 9$ are inequalities which are true, whereas $7 \neq 7$, $5 > 9$, and $7 < 4$ are inequalities which are false.

Consider the inequalities $8 > 5$ and $9 > 2$. Since the same inequality symbol of order is used in the two inequalities, we say they are *inequalities of the same order*. Also, $a < b$ and $c < d$ are inequalities of the same order.

Consider the inequalities $10 > 9$ and $6 < 8$. Since different inequality symbols of order are used in the two inequalities, we say they are *inequalities of the opposite order*. Also, $a > b$ and $c < d$ are inequalities of the opposite order.

Recall that the measure of a line segment is a real number. Hence, the inequality $AB > CD$ indicates that the number of a certain linear unit contained in line segment $\overline{AB}$ is greater than the number of the same linear unit contained in line segment $\overline{CD}$. Also, recall that the degree measure of an angle is a real number. Hence, the inequality $m\angle A > m\angle B$ indicates that the number of degrees contained in angle A is greater than the number of degrees contained in angle B.

In the discussion of inequalities involving geometric situations, we will simplify the language used in statements in the following manner:

Instead of saying that the measure of one line segment is greater than the measure of a second line segment, we will say that the first line segment is greater than the second line segment. We will follow a similar practice in dealing with lengths of arcs.

Instead of saying that the measure of one angle is greater than the measure of a second angle, we will say that the first angle is greater than the second angle.

EXERCISES

In 1–8, state whether the inequality is true or false.

1. $8 + 7 \neq 12$ **2.** $8 - 3 \neq 5$ **3.** $8 \times 0 \neq 7 \times 0$

4. $5 + 6 \neq 6 + 5$ **5.** $6 \times 1 \neq 6 \div 1$ **6.** $\frac{1}{4} + \frac{1}{4} \neq \frac{1}{4} \times \frac{1}{4}$

7. The product of 6 and 4 is not equal to the sum of 6 and 4.

8. The sum of 10 and 30 is not equal to 60 decreased by 20.

In 9–14, write the symbol $>$ or the symbol $<$ between each pair of numbers so that the resulting statement will be true.

9. 5 7 **10.** 12 8 **11.** 1 4 **12.** 15 20 **13.** 9 15 **14.** 25 17

In 15–20, express the inequality in words.

15. $25 > 17$ **16.** $16 < 22$ **17.** $9 + 4 > 8 + 2$

18. $7 - 3 < 13 - 4$ **19.** $3 \times 0 < 3 + 0$ **20.** $9 + 1 > 9 \times 1$

In 21–23, state whether the inequality is true or false.

21. $3 \times 5 < 4 + 6$ **22.** $6 \times 0 > 3 + 1$ **23.** $10 \times 1 > 20 \times 1$

In 24–27, state whether the two inequalities are of the same order or of opposite order.

24. $a > b$, $c > d$ **25.** $x < y$, $r > s$

26. $AB < DE$, $BC < EF$ **27.** $m\angle A > m\angle B$, $m\angle C < m\angle D$

2. Using Inequality Postulates in Proving Conclusions

We will now study postulates which involve unequal quantities, and we will learn how to use these postulates in proving conclusions.

Uniqueness of Order Postulate (The Trichotomy Postulate)

***Postulate 47.* Given any two quantities, exactly one of the following relations is true:**

1. **The first quantity is less than the second.**
2. **The first quantity is equal to the second.**
3. **The first quantity is greater than the second.**

Thus, if a and b are real numbers, then exactly one of the following relations is true: $a < b$, $a = b$, $a > b$.

Transitivity Property of Inequality

Postulate 48. **If the first of three quantities is greater than the second and the second is greater than the third, then the first is greater than the third.**

Thus, if a, b, and c are real numbers such that $a > b$ and $b > c$, then $a > c$.

EXAMPLE 1. If $10 > 8$ and $8 > 5$, then $10 > 5$.

EXAMPLE 2. (Fig. 9–1) If $BA > BD$ and $BD > BC$, then $BA > BC$.

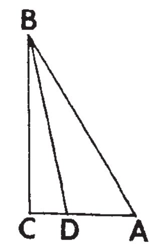
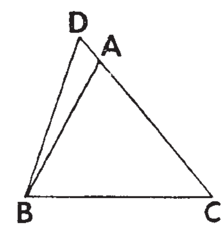


Fig. 9–1

MODEL PROBLEM

Given: $m\angle DBC > m\angle ABC$.
$m\angle ABC > m\angle ACB$.

To prove: $m\angle DBC > m\angle ACB$.



Proof:

Statements	*Reasons*
1. $m\angle DBC > m\angle ABC$.	1. Given.
2. $m\angle ABC > m\angle ACB$.	2. Given.
3. $m\angle DBC > m\angle ACB$.	3. The transitive property of inequality: If the first of three quantities is greater than the second and the second is greater than the third, then the first is greater than the third.

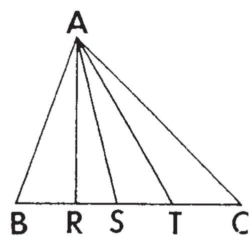
EXERCISES

In 1–3, state whether the conclusion is correct. Justify your answer with a property of inequality.

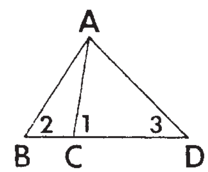
1. If Harry is older than Sam and Sam is older than Bill, then Harry is older than Bill.

2. If Sue weighs more than Marian and Marian weighs more than Roberta, then Sue weighs more than Roberta.

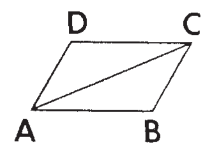
3. If a plane is traveling faster than a car and the car is traveling faster than a ship, then the plane is traveling faster than the ship.



Ex. 4



Ex. 5



Ex. 6

4. *Given:* $AT > AS$, $AS > AR$.
Prove: $AT > AR$.

5. *Given:* $m\angle 1 > m\angle 2$, $m\angle 2 > m\angle 3$.
Prove: $m\angle 1 > m\angle 3$.

6. *Given:* In $\square ABCD$, $AC > AB$, $AB > BC$.
Prove: $AC > BC$.

Substitution Postulate for Inequalities

***Postulate 49.* A quantity may be substituted for its equal in any inequality.**

Thus, if a, b, and c are real numbers such that $a > b$ and $c = b$, then $a > c$.

EXAMPLE 1. If $x > 7$
and $y = x$,
then $y > 7$.

EXAMPLE 2. (Fig. 9–2) If $m\angle 3 > m\angle 1$
and $m\angle 2 = m\angle 1$,
then $m\angle 3 > m\angle 2$.

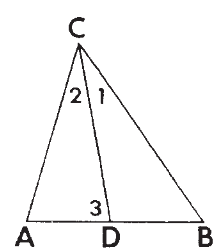
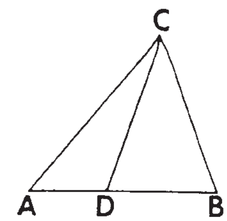


Fig. 9–2

MODEL PROBLEM

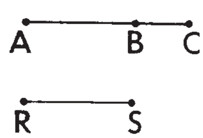
Given: $CB < CA$.
$CD = CB$.

To prove: $CD < CA$.

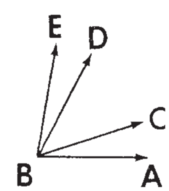


Proof: *Statements*	*Reasons*
1. $CB < CA$.	1. Given.
2. $CD = CB$.	2. Given.
3. $CD < CA$.	3. Substitution postulate for inequalities: A quantity may be substituted for its equal in any inequality.

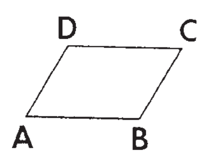
EXERCISES



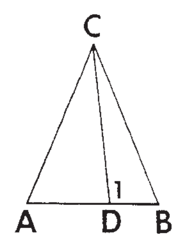
Ex. 1



Ex. 2



Ex. 3



Ex. 4

1. *Prove:* If $AC > AB$ and $RS = AB$, then $AC > RS$.

2. *Prove:* If $m\angle ABC < m\angle CBD$ and also if $m\angle ABC = m\angle DBE$, then $m\angle DBE < m\angle CBD$.

3. *Prove:* If $ABCD$ is a parallelogram and $AB > BC$, then $DC > BC$.

4. *Prove:* If $\triangle ABC$ is an isosceles triangle in which $\overline{AC} \cong \overline{CB}$ and $m\angle 1 > m\angle A$, then $m\angle 1 > m\angle B$.

Postulate Relating a Whole Quantity and Its Parts

***Postulate 50.* A whole quantity is greater than any of its parts.**

Thus, if a, b, and c are positive numbers such that $a = b + c$, then $a > b$ and $a > c$.

Fig. 9–3

Example 1. (Fig. 9–3) The measure of the whole line segment $\overline{AB}$ is greater than the measure of its part $\overline{AC}$ and is also greater than the measure of its part $\overline{CB}$, or $AB > AC$ and $AB > CB$. Segments $\overline{AC}$ and $\overline{CB}$ are referred to as parts of segment $\overline{AB}$.

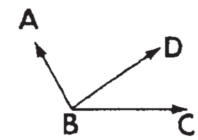


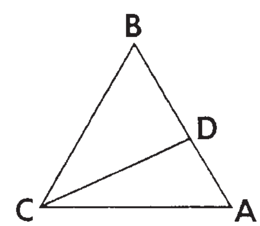
Fig. 9–4

Example 2. (Fig. 9–4) The measure of the whole $\angle ABC$ is greater than the measure of its part $\angle ABD$ and is also greater than the measure of its part $\angle CBD$, or $m\angle ABC > m\angle ABD$ and $m\angle ABC > m\angle CBD$. $\angle ABD$ and $\angle CBD$ are referred to as parts of $\angle ABC$.

EXERCISES

In 1–4, which refer to the figure, state whether the inequality is a true statement or a false statement. Justify your answer with an inequality postulate.

1. $m\angle ACB > m\angle ACD$.
2. $AB > BD$.
3. $m\angle ACB > m\angle BCD$.
4. $AB > AD$.



Ex. 1–4

Postulate Involving Addition of Equal Quantities and Unequal Quantities

***Postulate 51.* If equal quantities are added to unequal quantities, the sums are unequal in the same order.**

Thus, if a, b, c, and d are real numbers such that $a > b$, and $c = d$, then $a + c > b + d$.

Example 1.

$$\begin{aligned} \text{If } 5 &< 8 \\ \text{and } 4 &= 4 \\ \text{then } 5 + 4 &< 8 + 4, \\ \text{or } 9 &< 12. \end{aligned}$$

Example 2. (Fig. 9–5)

$$\begin{aligned} \text{If } AB &> CD \\ \text{and } BE &= DF, \\ \text{then } AB + BE &> CD + DF, \\ \text{or } AE &> CF. \end{aligned}$$

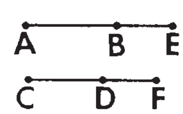
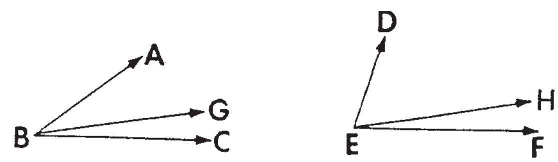


Fig. 9–5

MODEL PROBLEM

Given: $m\angle ABG < m\angle DEH$.
$m\angle GBC = m\angle HEF$.

To prove: $m\angle ABC < m\angle DEF$.

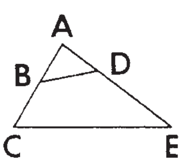


Proof:

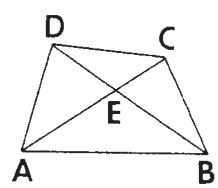
Statements	*Reasons*
1. $m\angle ABG < m\angle DEH$.	1. Given.
2. $m\angle GBC = m\angle HEF$.	2. Given.
3. $m\angle ABG + m\angle GBC < m\angle DEH + m\angle HEF$, or $m\angle ABC < m\angle DEF$.	3. If equal quantities are added to unequal quantities, the sums are unequal in the same order.

EXERCISES

1. *Prove:* If $10 > 7$, then $18 > 15$.
2. *Prove:* If $4 < 14$, then $15 < 25$.
3. *Prove:* If $x - 3 > 12$, then $x > 15$.
4. *Prove:* If $y - 9 < 5$, then $y < 14$.



Ex. 5



Ex. 6

5. *Prove:* If $AB = AD$ and $BC < DE$, then $AC < AE$.

6. *Prove:* If $m\angle DAC > m\angle DBC$ and $\overline{AE} \cong \overline{EB}$, then $m\angle DAB > m\angle CBA$.

Postulate Involving Addition of Unequal Quantities of the Same Order

Postulate 52. **If unequal quantities are added to unequal quantities of the same order, the sums are unequal in the same order.**

Thus, if a, b, c, and d are real numbers such that $a > b$ and $c > d$, then $a + c > b + d$.

EXAMPLE 1. If $6 > 4$
and $8 > 5$,
then $6 + 8 > 4 + 5$,
or $14 > 9$.

EXAMPLE 2. (Fig. 9–6) If $CD < AB$
and $DF < BE$,
then $CD + DF < AB + BE$,
or $CF < AE$.

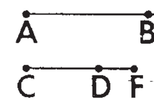
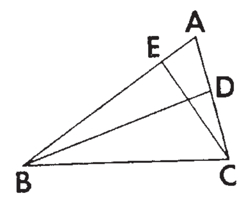


Fig. 9–6

MODEL PROBLEM

Given: $m\angle BCE > m\angle DBC$.
$m\angle ACE > m\angle ABD$.

To prove: $m\angle ACB > m\angle ABC$.



Proof:

Statements	*Reasons*
1. $m\angle BCE > m\angle DBC$.	1. Given.
2. $m\angle ACE > m\angle ABD$.	2. Given.
3. $m\angle BCE + m\angle ACE > m\angle DBC + m\angle ABD$, or $m\angle ACB > m\angle ABC$.	3. If unequal quantities are added to unequal quantities of the same order, the sums are unequal in the same order.

EXERCISES

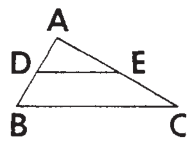
In 1 and 2 state whether the conclusion is correct. Justify your answer with an inequality postulate.

1. If Sidney weighs more than Ruth, and Jack weighs more than Martha, then Sidney and Jack together weigh more than Ruth and Martha together.

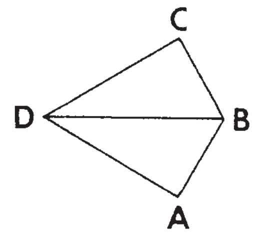
2. If Hilda has less money than Craig, and Rose has less money than Norman, then Hilda and Rose together have less money than Craig and Norman together.

3. *Prove:* If $8 > 6$, then $12 > 9$.

4. *Prove:* If $15 < 21$, then $18 < 27$.



Ex. 5



Ex. 6

5. *Prove:* If $AE > AD$ and $EC > DB$, then $AC > AB$.

6. *Prove:* If $m\angle CDB < m\angle CBD$ and $m\angle ADB < m\angle ABD$, then $m\angle CDA < m\angle CBA$.

Postulate Involving Subtraction of Equal Quantities From Unequal Quantities

Postulate 53. **If equal quantities are subtracted from unequal quantities, the differences are unequal in the same order.**

Thus, if a, b, c, and d are real numbers such that $a > b$ and $c = d$, then $a - c > b - d$.

EXAMPLE 1. If $15 > 12$
and $5 = 5$,
then $15 - 5 > 12 - 5$,
or $10 > 7$.

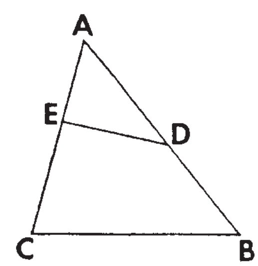


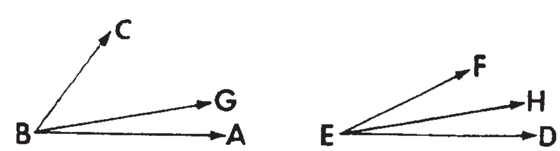
Fig. 9-7

EXAMPLE 2. (Fig. 9-7) If $AC < AB$
and $EC = DB$,
then $AC - EC < AB - DB$,
or $AE < AD$.

MODEL PROBLEM

Given: $m\angle ABC > m\angle DEF$.
$m\angle ABG = m\angle DEH$.

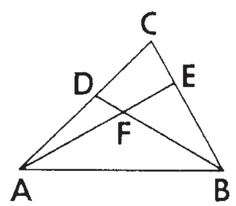
To prove: $m\angle GBC > m\angle HEF$.



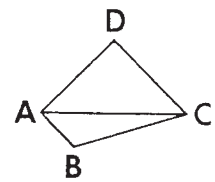
Proof: *Statements*	*Reasons*
1. $m\angle ABC > m\angle DEF$.	1. Given.
2. $m\angle ABG = m\angle DEH$.	2. Given.
3. $m\angle ABC - m\angle ABG > m\angle DEF - m\angle DEH$, or $m\angle GBC > m\angle HEF$.	3. If equal quantities are subtracted from unequal quantities, the differences are unequal in the same order.

EXERCISES

1. *Prove:* If $18 > 12$, then $15 > 9$.
2. *Prove:* If $9 < 13$, then $7 < 11$.
3. *Prove:* If $x + 5 > 13$, then $x > 8$.
4. *Prove:* If $y + 3 < 9$, then $y < 6$.



Ex. 5



Ex. 6

5. *Prove:* If $AE > BD$ and $AF = BF$, then $FE > FD$.

6. *Prove:* If $m\angle DCB < m\angle DAB$ and $\overline{AD} \cong \overline{DC}$, then $m\angle ACB < m\angle CAB$.

Postulate Involving Subtraction of Unequal Quantities From Equal Quantities

Postulate 54. **If unequal quantities are subtracted from equal quantities, the differences are unequal in the opposite order.**

Thus, if a, b, c, and d are real numbers such that $a = b$ and $c < d$, then $a - c > b - d$.

EXAMPLE 1.

$$\begin{aligned} &\text{If } 10 = 10 \\ &\text{and } 8 > 4, \\ &\text{then } 10 - 8 < 10 - 4, \\ &\text{or } 2 < 6. \end{aligned}$$

EXAMPLE 2. (Fig. 9–8)

$$\begin{aligned} &\text{If } CD = AB \\ &\text{and } CF < AE, \\ &\text{then } CD - CF > AB - AE, \\ &\text{or } FD > EB. \end{aligned}$$

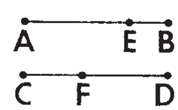


Fig. 9–8

MODEL PROBLEM

Given: $m\angle DEF = m\angle ABC$.
$m\angle DEH > m\angle ABG$.

To prove: $m\angle HEF < m\angle GBC$.

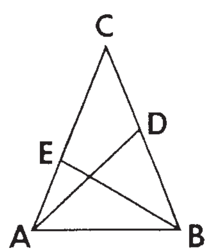
[The proof is given on the next page.]

Proof: *Statements*	*Reasons*
1. $m\angle DEF = m\angle ABC$.	1. Given.
2. $m\angle DEH > m\angle ABG$.	2. Given.
3. $m\angle DEF - m\angle DEH < m\angle ABC - m\angle ABG$, or $m\angle HEF < m\angle GBC$.	3. If unequal quantities are subtracted from equal quantities, the differences are unequal in the opposite order.

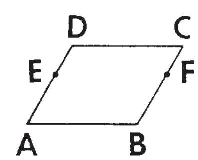
EXERCISES

1. *Prove:* If $90 = 90$ and $x > y$, then $(90 - x) < (90 - y)$.

2. *Prove:* If $x < y$, then $(180 - x) > (180 - y)$.



Ex. 3



Ex. 4

3. *Prove:* If $m\angle CAB = m\angle CBA$ and $m\angle BAD > m\angle ABE$, then $m\angle CAD < m\angle CBE$.

4. *Prove:* If $ABCD$ is a parallelogram and $ED < BF$, then $AE > FC$.

Postulate Involving Multiplication of Unequal Quantities by Equal Positive Quantities

Postulate 55. **If unequal quantities are multiplied by equal positive quantities, the products are unequal in the same order. [A special case of this postulate is: Doubles of unequal quantities are unequal in the same order.]**

Thus, if a, b, c, and d are positive numbers such that $a > b$ and $c = d$, then $ac > bd$.

EXAMPLE 1.

$$\begin{aligned} &\text{If } 9 > 7 \\ &\text{and } 4 = 4, \\ &\text{then } 4 \times 9 > 4 \times 7, \\ &\text{or } 36 > 28. \end{aligned}$$

EXAMPLE 2. (Fig. 9–9) If $AB > DE$ and $AC = 2AB$, $DF = 2DE$ then $AC > DF$ because doubles of unequal quantities are unequal in the same order.

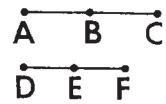
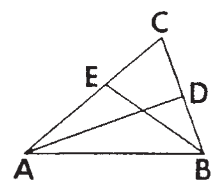


Fig. 9–9

MODEL PROBLEM



Given: $m\angle DAB < m\angle EBA$.
$\overline{AD}$ bisects $\angle CAB$, $\overline{BE}$ bisects $\angle CBA$.

To prove: $m\angle CAB < m\angle CBA$.

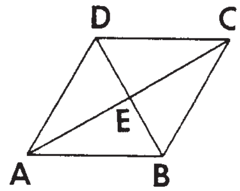
Proof: *Statements*	*Reasons*
1. $m\angle DAB < m\angle EBA$.	1. Given.
2. $2m\angle DAB < 2m\angle EBA$.	2. Doubles of unequal quantities are unequal in the same order.
3. $\overline{AD}$ bisects $\angle CAB$, $\overline{BE}$ bisects $\angle CBA$.	3. Given.
4. $m\angle CAB = 2m\angle DAB$, $m\angle CBA = 2m\angle EBA$.	4. A bisector of an angle divides the angle into two congruent angles.
5. $m\angle CAB < m\angle CBA$.	5. A quantity may be substituted for its equal in any inequality.

EXERCISES

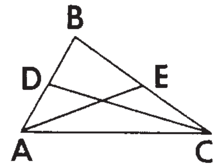
1. *Prove:* If $\frac{x}{2} > 6$, then $x > 12$.

2. *Prove:* If $\frac{y}{5} < 4$, then $y < 20$.

3. If $a > b$ and c is a positive number, which of the following relationships is *not* true?
(1) $ac > bc$ (2) $c - a > c - b$ (3) $a + c > b + c$ (4) $a - c > b - c$



Ex. 4



Ex. 5

4. *Prove:* If $m\angle DBA > m\angle CAB$, $m\angle CBA = 2m\angle DBA$, and $m\angle DAB = 2m\angle CAB$, then $m\angle CBA > m\angle DAB$.
5. *Prove:* If $BD < BE$, D is the midpoint of $\overline{AB}$, and E is the midpoint of $\overline{BC}$, then $BA < BC$.

Postulate Involving Division of Unequal Quantities by Equal Positive Quantities

Postulate 56. **If unequal quantities are divided by equal positive quantities, the quotients are unequal in the same order. [A special case of this postulate is: Halves of unequal quantities are unequal in the same order.]**

Thus, if a, b, c, and d are positive numbers such that $a > b$ and $c = d$, then $\frac{a}{c} > \frac{b}{d}$.

EXAMPLE 1. If $12 > 8$
and $4 = 4$,
then $\frac{12}{4} > \frac{8}{4}$,
or $3 > 2$.

EXAMPLE 2. (Fig. 9–10) If $AC < AB$
and $AE = \frac{1}{2}AC$,
$AD = \frac{1}{2}AB$,
then $AE < AD$, because halves of unequal quantities are unequal in the same order.

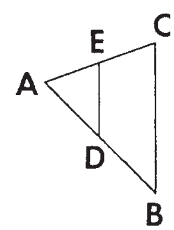
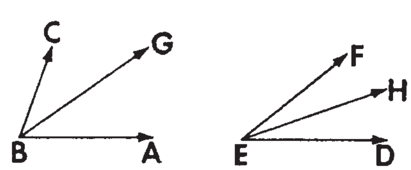


Fig. 9–10

MODEL PROBLEM

Given: $m\angle ABC > m\angle DEF$.
$\overrightarrow{BG}$ bisects $\angle ABC$.
$\overrightarrow{EH}$ bisects $\angle DEF$.
To prove: $m\angle ABG > m\angle DEH$.

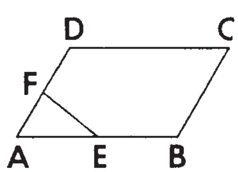


Proof: *Statements*	*Reasons*
1. $m\angle ABC > m\angle DEF$.	1. Given.
2. $\dfrac{m\angle ABC}{2} > \dfrac{m\angle DEF}{2}$, or $\frac{1}{2}m\angle ABC > \frac{1}{2}m\angle DEF$.	2. Halves of unequal quantities are unequal in the same order.
3. $\overrightarrow{BG}$ bisects $\angle ABC$, $\overrightarrow{EH}$ bisects $\angle DEF$.	3. Given.
4. $m\angle ABG = \frac{1}{2}m\angle ABC$, $m\angle DEH = \frac{1}{2}m\angle DEF$.	4. A bisector of an angle divides the angle into two congruent angles.
5. $m\angle ABG > m\angle DEH$.	5. A quantity may be substituted for its equal in any inequality.

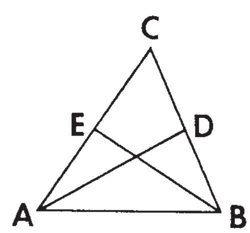
EXERCISES

1. *Prove:* If $2x > 14$, then $x > 7$.

2. *Prove:* If $5y < 40$, then $y < 8$.



Ex. 3



Ex. 4

3. *Prove:* If $AB > AD$, $AE = \frac{1}{2}AB$, and $AF = \frac{1}{2}AD$, then $AE > AF$.

4. *Prove:* If $m\angle CAB < m\angle CBA$, $\overline{AD}$ bisects $\angle CAB$, and $\overline{BE}$ bisects $\angle CBA$, then $m\angle DAB < m\angle EBA$.

Postulate Involving Positive Powers and Positive Roots of Unequal Positive Quantities

Postulate 57. **Like positive integral powers and like positive integral roots of unequal positive quantities are unequal in the same order.**

Thus, if a, b, and n are positive integers such that $a > b$, then $a^n > b^n$, and $\sqrt[n]{a} > \sqrt[n]{b}$.

EXAMPLE 1. If $y > 7$, then $y^2 > 49$.

EXAMPLE 2. If $x^2 < 25$, then $x < 5$, when x is a positive number.

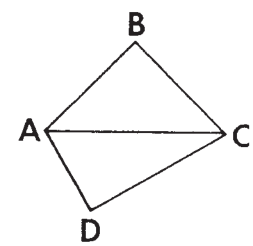
3. Additional Exercises Using Inequality Postulates in Proving Conclusions

Now let us see how the inequality postulates we have just studied can be combined with other postulates, theorems, and corollaries in proving conclusions.

MODEL PROBLEMS

1. *Given:* Isosceles triangle ABC with $\overline{AB} \cong \overline{BC}$. $m\angle BAD > m\angle BCD$.

To prove: $m\angle CAD > m\angle ACD$.



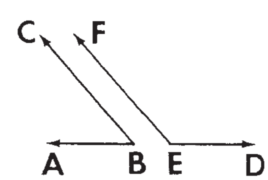
Proof: *Statements*	*Reasons*
1. $m\angle BAD > m\angle BCD$.	1. Given.
2. $\overline{AB} \cong \overline{BC}$.	2. Given.
3. $m\angle BAC = m\angle BCA$.	3. If two sides of a triangle are congruent, the angles opposite these sides are congruent.
4. $m\angle BAD - m\angle BAC > m\angle BCD - m\angle BCA$, or $m\angle CAD > m\angle ACD$.	4. If equal quantities are subtracted from unequal quantities, the differences are unequal in the same order.

2. Prove that the supplement of an obtuse angle is an acute angle.

Given: $\angle DEF$ is an obtuse angle. $\angle ABC$ is supplementary to $\angle DEF$.

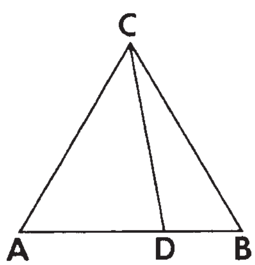
To prove: $\angle ABC$ is an acute angle.

Plan: To prove that $\angle ABC$ is an acute angle, prove that $m\angle ABC < 90$.

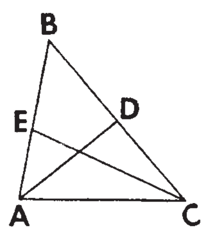


Proof: *Statements*	*Reasons*
1. $\angle DEF$ is an obtuse angle, $\angle ABC$ is supplementary to $\angle DEF$.	1. Given.
2. $m\angle ABC + m\angle DEF = 180$.	2. The sum of the measures of two angles which are supplementary is 180.
3. $m\angle DEF > 90$.	3. The measure of an obtuse angle is greater than 90 and less than 180.
4. $m\angle ABC < 90$.	4. If unequal quantities are subtracted from equal quantities, the differences are unequal in the opposite order.
5. $\angle ABC$ is an acute angle.	5. An angle whose measure is less than 90 and greater than 0 is an acute angle.

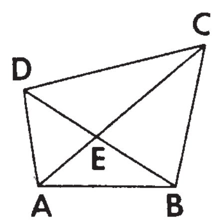
EXERCISES



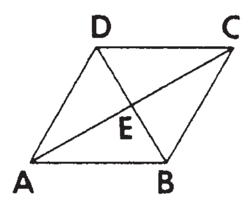
Ex. 1



Ex. 2



Ex. 3



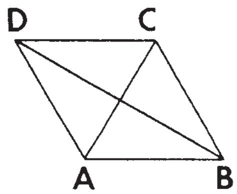
Ex. 4

1. *Given:* Isosceles triangle ABC with $\overline{AC} \cong \overline{CB}$, $m\angle CDB > m\angle CAD$.
Prove: $m\angle CDB > m\angle CBA$.

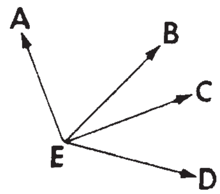
2. *Given:* $m\angle BAC > m\angle BCA$, $\overline{AD}$ bisects $\angle BAC$, $\overline{CE}$ bisects $\angle BCA$.
Prove: $m\angle DAC > m\angle ECA$.

3. *Given:* Quadrilateral $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$, $AE + EB > AB$, $CE + ED > DC$.
Prove: $AC + BD > AB + DC$.

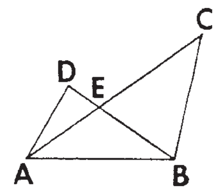
4. *Given:* In parallelogram $ABCD$, $AE > BE$.
Prove: $AC > BD$.



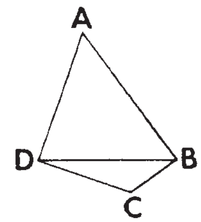
Ex. 5



Ex. 6



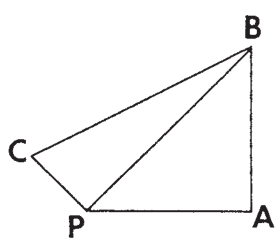
Ex. 7



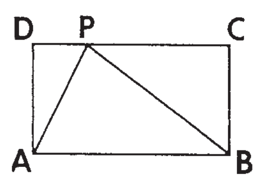
Ex. 8

5. *Given:* $ABCD$ is a rhombus, $m\angle CBA < m\angle DAB$.
   Prove: $m\angle DBA < m\angle CAB$.
6. *Given:* $m\angle AEB > m\angle CED$.
   Prove: $m\angle AEC > m\angle DEB$.
7. *Given:* $m\angle CBA > m\angle DAB$, $\overline{AE} \cong \overline{EB}$.
   Prove: $m\angle CBD > m\angle DAE$.
8. *Given:* $\overline{AB} \perp \overline{BC}$, $\overline{AD} \perp \overline{DC}$, $m\angle ADB > m\angle ABD$.
   Prove: $m\angle BDC < m\angle DBC$.
9. *Given:* $\overleftrightarrow{ABC}$ and $\overleftrightarrow{EFG}$.
   $m\angle HFG < m\angle DBC$.
   Prove: $m\angle HFE > m\angle DBA$.
10. *Prove:* The supplement of an acute angle is an obtuse angle.

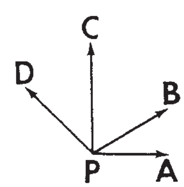
Ex. 9



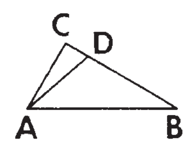
Ex. 11



Ex. 12

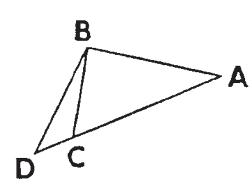


Ex. 13



Ex. 14

11. *Given:* $m\angle CPB > m\angle CBP$, $\overline{AP} \cong \overline{AB}$.
    Prove: $m\angle CPA > m\angle CBA$.
12. *Given:* $ABCD$ is a rectangle, $m\angle PAB > m\angle PBA$.
    Prove: $m\angle PAD < m\angle PBC$.
13. *Given:* $m\angle CPB > m\angle BPA$.
    Prove: (*a*) $m\angle DPB > m\angle CPB$.
    (*b*) $m\angle DPB > m\angle BPA$.
14. *Given:* $m\angle DAB > m\angle DBA$.
    Prove: $m\angle CAB > m\angle DBA$.
15. *Given:* $m\angle ABC > m\angle ADB$.
    Prove: $m\angle ABD > m\angle ADB$.



Ex. 15

16. In $\triangle DEF$ and $\triangle D'E'F'$, $m\angle D = m\angle D'$, $m\angle E > m\angle E'$.
Prove: $m\angle F < m\angle F'$.

17. *Prove:* If $m\angle A$ is greater than $m\angle B$, then the measure of the complement of $\angle A$ is less than the measure of the complement of $\angle B$.

18. *Prove:* If $m\angle R$ is less than $m\angle S$, then the measure of the supplement of $\angle R$ is greater than the measure of the supplement of $\angle S$.

4. Inequalities in a Triangle

We have already studied the postulate "The shortest path between two points is the line segment joining the two points." Using this postulate, we can readily show:

Theorem 129. **The sum of the lengths of two sides of a triangle is greater than the length of the third side.**

In $\triangle ABC$ (Fig. 9–11), the sum of the lengths of two sides is greater than the length of the third side, or $AC + CB > BA$, $CB + BA > AC$, and also $BA + AC > CB$.

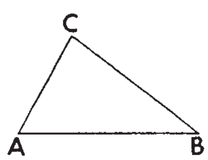


Fig. 9–11

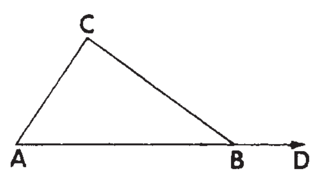


Fig. 9–12

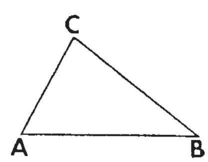


Fig. 9–13

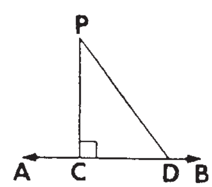


Fig. 9–14

Theorem 130. **The measure of an exterior angle of a triangle is greater than the measure of either nonadjacent interior angle.**

In Fig. 9–12, if $\angle CBD$ is an exterior angle of triangle ABC, then $m\angle CBD > m\angle A$ and $m\angle CBD > m\angle C$.

Theorem 131. **If two sides of a triangle are unequal, the angles opposite these sides are unequal and the greater angle lies opposite the greater side.**

In $\triangle ABC$ (Fig. 9–13), if CB is greater than CA, then the measure of the angle opposite $\overline{CB}$ is greater than the measure of the angle opposite $\overline{CA}$, or $m\angle A > m\angle B$.

Theorem 132. **If two angles of a triangle are unequal, the sides opposite these angles are unequal and the greater side lies opposite the greater angle.**

In $\triangle ABC$ (Fig. 9–13), if $m\angle A$ is greater than $m\angle B$, then the measure of the side opposite $\angle A$ is greater than the measure of the side opposite $\angle B$, or $CB > CA$.

Corollary T132–1. **The shortest line segment that can be drawn joining a point not on a given line to the given line is the line segment drawn perpendicular to the given line from the given point.**

In Fig. 9–14, if $\overline{PC}$ is perpendicular to $\overleftrightarrow{AB}$, then segment $\overline{PC}$ is shorter than any other segment that can be drawn joining P and a point in $\overline{AB}$, such as $\overline{PD}$.

Methods of Proof:

1. To prove that the length of one line segment is greater than the length of a second line segment, show that the two segments are two sides in a triangle and that the measure of the angle opposite the first segment is greater than the measure of the angle opposite the second segment.
2. To prove that the measure of one angle is greater than the measure of a second angle:
   a. show that they are angles of a triangle and that the length of the side opposite the first angle is greater than the length of the side opposite the second angle.

   OR

   b. show that the first angle is an exterior angle of a triangle in which the second angle is a nonadjacent interior angle.

MODEL PROBLEMS

1. If the lengths of two sides of a triangle are 10 and 14, the length of the third side may be (*a*) 2 (*b*) 4 (*c*) 22 (*d*) 24.

Solution: In a triangle, the sum of the lengths of any two sides must be greater than the length of the third side. We can therefore discover if three lengths can be the sides of a triangle by determining whether or not the sum of the two shorter lengths is greater than the third length.

a. 2 cannot be the third side because $2 + 10$ is not > 14.

b. 4 cannot be the third side because $4 + 10$ is not > 14.

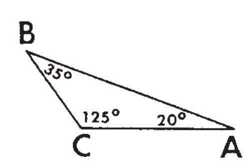
c. 22 can be the third side because $10 + 14 > 22$.

d. 24 cannot be the third side because $10 + 14$ is not > 24.

Answer: The third side may be 22.

2. In $\triangle ABC$, $m\angle C = 125$, $m\angle B = 35$. Which is the shortest side of the triangle?

Solution:

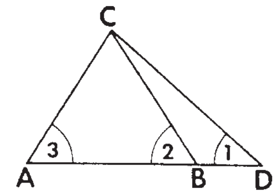


1. Since $m\angle B = 35$ and $m\angle C = 125$,
   $m\angle A = 180 - (125 + 35) = 180 - 160 = 20$.

2. Since the *shortest* side of a triangle is opposite the smallest angle, the shortest side of $\triangle ABC$ is $\overline{BC}$, which is opposite the smallest angle, $\angle A$.

Answer: $\overline{BC}$ is the shortest side of $\triangle ABC$.

3. In isosceles $\triangle ABC$, with $\overline{AC} \cong \overline{CB}$, base $\overline{AB}$ is extended to D and $\overline{CD}$ is drawn. *Prove:* $CD > CA$.



Given: $\overline{AC} \cong \overline{CB}$.
$\overline{AB}$ is extended to D.
$\overline{CD}$ is drawn.

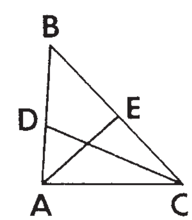
To prove: $CD > CA$.

Plan: To prove that $CD > CA$, show that $\overline{CD}$ and $\overline{CA}$ are in $\triangle ACD$ with $m\angle 3$, which is opposite $\overline{CD}$, greater than $m\angle 1$, which is opposite $\overline{CA}$.

Proof:

Statements	*Reasons*
1. In $\triangle CBD$, $m\angle 2 > m\angle 1$.	1. The measure of an exterior angle of a triangle is greater than the measure of either nonadjacent interior angle.
2. $\overline{AC} \cong \overline{CB}$.	2. Given.
3. $\angle 3 \cong \angle 2$, or $m\angle 3 = m\angle 2$.	3. Base angles of an isosceles triangle are congruent.
4. $m\angle 3 > m\angle 1$.	4. A quantity may be substituted for its equal in any inequality.
5. $CD > CA$.	5. If two angles of a triangle are unequal, the sides opposite these angles are unequal, and the greater side lies opposite the greater angle.

4. *Given:* In $\triangle ABC$, $BC > BA$.
$\overline{CD}$ bisects $\angle BCA$.
$\overline{AE}$ bisects $\angle BAC$.



To prove: $m\angle EAC > m\angle DCA$.

Plan: To prove $m\angle EAC > m\angle DCA$, show that in $\triangle ABC$, $m\angle BAC > m\angle BCA$ because $BC > BA$; and then show that $m\angle EAC$, which is $\frac{1}{2}m\angle BAC$, is greater than $m\angle DCA$, which is $\frac{1}{2}m\angle BCA$.

[The proof is given on the next page.]

Proof: *Statements*	*Reasons*
1. In $\triangle ABC$, $BC > BA$.	1. Given.
2. $m\angle BAC > m\angle BCA$.	2. If two sides of a triangle are unequal, the angles opposite these sides are unequal, and the greater angle lies opposite the greater side.
3. $\overline{AE}$ bisects $\angle BAC$, $\overline{CD}$ bisects $\angle BCA$.	3. Given.
4. $m\angle EAC = \frac{1}{2}m\angle BAC$, $m\angle DCA = \frac{1}{2}m\angle BCA$.	4. A bisector of an angle divides the angle into two congruent angles.
5. $m\angle EAC > m\angle DCA$.	5. Halves of unequal quantities are unequal in the same order.

EXERCISES

1. Tell whether the given lengths may be the measures of the sides of a triangle:
a. 6 in., 4 in., 10 in. *b.* 6 in., 4 in., 12 in. *c.* 6 in., 4 in., 8 in.

2. Tell which of the following number triples may be used as the lengths of the sides of a triangle:
a. (7, 8, 9) *b.* (3, 5, 8) *c.* (8, 5, 2) *d.* (3, 10, 6) *e.* (6, 9, 10)

3. The lengths of two sides of a triangle are 3 in. and 6 in. The length of the third side may be:
a. 3 in. *b.* 6 in. *c.* 9 in. *d.* 12 in.

4. Which one of the following number triples can *not* represent the length units of the sides of a triangle?
a. (2, 3, 4) *b.* (3, 1, 1) *c.* (3, 4, 5) *d.* (3, 4, 4)

5. In triangle ABC, $AB = 8$, $BC = 10$, and $CA = 14$. Name the largest angle of triangle ABC.

6. In triangle ABC, angle C contains 60° and AB is greater than AC. Angle B contains (*a*) 60° (*b*) less than 60° (*c*) more than 60°.

7. In triangle ABC, $CA > CB$ and $m\angle B = 35$. Angle C is (*a*) an acute angle (*b*) a right angle (*c*) an obtuse angle.

8. In $\triangle ABC$, $m\angle c = 90$ and $m\angle B = 35$. Name the shortest side of this triangle.

9. In $\triangle ABC$, $m\angle A = 74$ and $m\angle B = 58$. Which is the longest side of the triangle?

10. If in $\triangle RST$, $m\angle R = 71$ and $m\angle S = 37$, then (*a*) $ST > RS$ (*b*) $RS > RT$ (*c*) $RS = ST$ (*d*) $RT > ST$.

11. In $\triangle RST$, an exterior angle at R contains 120°. If $m\angle S > m\angle T$, the longest side of the triangle is (*a*) $\overline{RS}$ (*b*) $\overline{ST}$ (*c*) $\overline{TR}$.

12. In $\triangle RST$, $m\angle T = 60$ and an exterior angle at R contains 130°. Which is the longest side of the triangle?

13. In $\triangle RST$, $\angle R$ is obtuse and $m\angle S = 50$. Name the shortest side of the triangle.

14. In $\triangle RST$, $m\angle R > m\angle S$ and the bisectors of $\angle R$ and $\angle S$ meet in P. PS is (*a*) equal to PR (*b*) less than PR (*c*) greater than PR.

15. In $\triangle ABC$, if an exterior angle at C contains 110°, then (*a*) $m\angle A < 110$ (*b*) $m\angle A = 110$ (*c*) $m\angle A > 110$.

16. In $\triangle DEF$, if an exterior angle at D contains 90°, then $\angle E$ is (*a*) an acute angle (*b*) a right angle (*c*) an obtuse angle.

17. Given $\triangle RST$ with side $\overline{RT}$ extended through T to W. Then for any $\triangle RST$ (*a*) $m\angle WTS < m\angle STR$ (*b*) $m\angle WTS > m\angle STR$ (*c*) $m\angle WTS > m\angle R$ (*d*) $m\angle WTS < m\angle R$.

18. In isosceles triangle ABC, $\overline{AC} \cong \overline{CB}$. If D is a point on the base $\overline{AB}$ lying between A and B, and $\overline{CD}$ is drawn, then (*a*) $AC > CD$ (*b*) $CD > AC$ (*c*) $m\angle A > m\angle ADC$ (*d*) $m\angle B > m\angle BDC$.

19. In isosceles triangle RQS, $\overline{QR} \cong \overline{QS}$. If $\overline{RS}$ is extended to point P and $\overline{QP}$ is drawn, it is always true that (*a*) $m\angle QRS > m\angle RQS$ (*b*) $m\angle QRS > m\angle RPQ$ (*c*) $m\angle SQP > m\angle RPQ$ (*d*) $m\angle SPQ > m\angle RQS$.

20. $\triangle DEF$ is an acute triangle. The shortest line segment drawn from D to $\overline{EF}$ is (*a*) the bisector of angle D (*b*) the altitude to $\overline{EF}$ (*c*) the median to $\overline{EF}$.

In 21–27, if the blank space is replaced by the word *always, sometimes,* or *never,* the resulting statement will be true. Select the word which will correctly complete each statement.

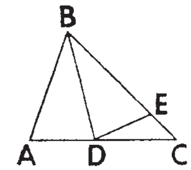
21. In triangle ABC, angle A contains more than 60°. Side $\overline{BC}$ is _______ the longest side of the triangle.

22. If one angle of a scalene triangle contains 60 degrees, the side opposite this angle is _______ the longest side of the triangle.

23. If one of the congruent sides of an isosceles triangle is longer than the base, then the measure of the angle opposite the base is _______ greater than 60.

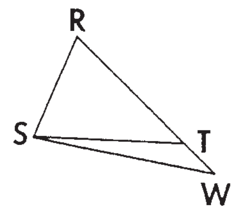
24. In triangle ABC, if AB is greater than AC, then $m\angle C$ is _______ greater than $m\angle B$.

25. The angles whose vertices are the endpoints of the longest side of a triangle are ______ acute angles.
26. If the three sides of a triangle are unequal in length, the altitude upon any side is ______ equal to the median to that side.
27. An altitude and a median are drawn from the same vertex of a triangle to the opposite side. The altitude is ______ greater than the median.
28. *Given:* $m\angle DEB = m\angle DAB$. Points B, E, and C are collinear.
Prove: $m\angle BAC > m\angle BCD$.



Ex. 28

29. *Given:* $m\angle RST > m\angle RTS$.
Prove: $m\angle RST > m\angle RWS$.



Ex. 29

30. In circle O, radii $\overline{OC}$ and $\overline{OD}$ and chord $\overline{CD}$ are drawn. Point T is taken on radius $\overline{OC}$ between O and C, and $\overline{TD}$ is drawn. Prove that $TD > TC$.
31. In circle O if $\overline{OBA}$ and $\overline{OC}$ are radii and $\overline{BC}$ is drawn, prove:
a. $OB + BC > OA$.
b. $BC > BA$.
32. In isosceles $\triangle ABC$, $\overline{CA} \cong \overline{CB}$. If D is a point on $\overline{AC}$ between A and C, prove that $DB > DA$.
33. Given $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$; D is any point between B and C on $\overline{BC}$; line segment $\overline{AD}$ is drawn. Prove that $AB > AD$.
34. In $\triangle ABC$, $\overline{AB} \cong \overline{AC}$. Prove that an exterior angle at B is an obtuse angle.
35. In $\triangle ABC$, the bisector of $\angle C$ meets $\overline{AB}$ in D. Prove that $CB > BD$.
36. In $\triangle ABC$, $AC > AB$. The bisector of $\angle B$ and the bisector of $\angle C$ intersect in D. Prove that $DC > DB$.
37. In $\triangle ABC$, $\overline{AB} \cong \overline{AC}$. If $\overline{AD}$ is the median to $\overline{BC}$, prove that $AB > AD$.
38. In $\triangle RST$, $\overline{SR} \cong \overline{ST}$. The bisector of $\angle R$ and the bisector of $\angle T$ intersect in P. Prove that $RT > PT$.
39. In parallelogram $ABCD$, $\overline{AD}$ is longer than $\overline{DC}$ and diagonal $\overline{AC}$ is drawn. Prove that $\overline{CA}$ does not bisect angle C.
40. $ABCD$ is a quadrilateral in which $AD > DC$ and $AB > BC$. Prove that $m\angle BCD > m\angle DAB$. [*Hint:* Draw $\overline{AC}$.]
41. In parallelogram $PQRS$, $PQ > QR$ and diagonal $\overline{PR}$ is drawn. Prove that $m\angle SPR > m\angle RPQ$.
42. In quadrilateral $PRST$, $\overline{PR} \cong \overline{RS}$ and $PT > ST$. Prove that $m\angle RST > m\angle RPT$.

43. In acute $\triangle ABC$, the altitude from B meets $\overline{AC}$ at D. If $m\angle ABD > m\angle CBD$, prove that $AB > BC$.

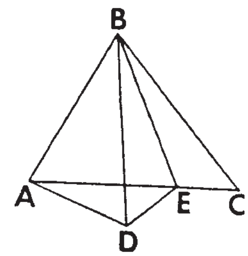
44. If median $\overline{AD}$ of $\triangle ABC$ is longer than $\overline{BD}$, prove that $m\angle BAC$ is less than the sum of $m\angle B$ and $m\angle C$.

45. Prove that the sum of the lengths of the diagonals of a quadrilateral is greater than the sum of the lengths of a pair of opposite sides.

46. In parallelogram $ABCD$, $\overline{AC}$ is the longer diagonal. Point P is taken on $\overline{AC}$ so that $AP = AB$. Prove that $BC > PC$.

47. In $\triangle ABC$, $AC > AB$. $\overline{CA}$ is extended through A to a point D and $\overline{BD}$ is drawn. Prove that $DC > DB$.

48. In the figure, $\overline{BC} \cong \overline{BD}$ and $\overline{BE}$ is the bisector of angle DBC. (*a*) Prove that $\overline{ED} \cong \overline{EC}$. (*b*) Using the fact that $\overline{ED} \cong \overline{EC}$, prove $AC > AD$.



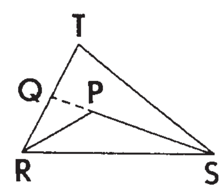
Ex. 48

49. In $\triangle ABC$, $\overline{AB} \cong \overline{AC}$. A line through B intersects $\overline{AC}$ at D. $\overline{BD}$ is extended through D to point E and $\overline{CE}$ is drawn. Prove that $BE > CE$.

50. *Prove:* The difference between the lengths of two sides of a triangle is less than the length of the third side.

51. *Prove:* The sum of the lengths of the three altitudes of a triangle is less than the perimeter of the triangle.

52. *Given:* Point P is in the interior of $\triangle RST$.
a. Prove: $RT + TS > RP + PS$. [*Hint:* Extend $\overline{SP}$ until it intersects $\overline{RT}$ at Q.]
b. Prove: $m\angle RPS > m\angle RTS$.

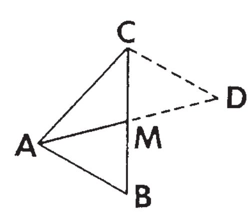


Ex. 52

53. Point P is in the interior of $\triangle RST$. Prove that the sum of the distances from P to the vertices of $\triangle RST$ is (*a*) less than the perimeter of $\triangle RST$ (*b*) greater than half the perimeter of $\triangle RST$. [*Hint:* In proving part *a*, use the fact proved in exercise 52*a*.]

54. *Prove:* The perimeter of a quadrilateral is greater than the sum of the lengths of its diagonals.

55. *Prove:* The length of a median drawn to a side of a triangle is less than one-half the sum of the lengths of the other two sides. [*Hint:* In $\triangle ABC$, extend median $\overline{AM}$ to point D so that $AM = MD$. Show that $\overline{CD} \cong \overline{AB}$. Use the fact that in $\triangle ACD$, $AC + CD > AD$.]



Ex. 55

56. *Prove:* The sum of the lengths of the medians of a triangle is less than the perimeter of the triangle.

5. Inequalities in Two Triangles

Theorem 133. **If two triangles have two sides of one congruent respectively to two sides of the other and the included angles are unequal, then the triangle which has the greater included angle has the greater third side.**

In $\triangle ABC$ and $\triangle A'B'C'$ (Fig. 9–15), if $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, and $m\angle B > m\angle B'$, then $AC > A'C'$.

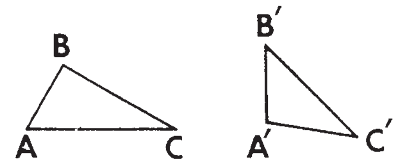


Fig. 9–15

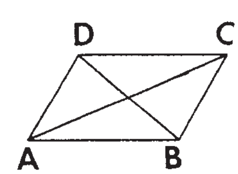
Theorem 134. **If two triangles have two sides of one congruent respectively to two sides of the other and the third sides are unequal, then the triangle which has the greater third side has the greater angle opposite this side.**

In $\triangle ABC$ and $\triangle A'B'C'$ (Fig. 9–15), if $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, and $AC > A'C'$, then $m\angle B > m\angle B'$.

Methods of Proof:

1. To prove that the lengths of two line segments which are not in the same triangle are unequal. show that the segments are sides in two triangles which have two sides of one congruent to two sides of the other and that the angles included between these sides are unequal.
2. To prove that the measures of two angles which are not in the same triangle are unequal, show that the angles are opposite unequal sides in two triangles which have two sides of one congruent to two sides of the other.

MODEL PROBLEM



Given: $ABCD$ is a parallelogram with diagonals $\overline{AC}$ and $\overline{BD}$. $\angle ABC$ is an obtuse angle.

To prove: $AC > BD$.

Plan: Since $\overline{AC}$ and $\overline{BD}$ are in different triangles ($\triangle ABC$ and $\triangle BAD$), show that two sides in $\triangle ABC$ ($\overline{BC}$ and $\overline{AB}$) are congruent to two sides in $\triangle BAD$ ($\overline{AD}$ and $\overline{AB}$), and that the measure of the included angle ABC is greater than the measure of the included angle BAD.

Proof: *Statements*	*Reasons*
1. $ABCD$ is a parallelogram.	1. Given.
2. $\overline{AD} \cong \overline{BC}$.	2. Both pairs of opposite sides of a parallelogram are congruent.
3. $\angle BAD$ is supplementary to $\angle ABC$.	3. The consecutive angles of a parallelogram are supplementary.
4. $\angle ABC$ is an obtuse angle.	4. Given.
5. $\angle BAD$ is an acute angle.	5. The supplement of an obtuse angle is an acute angle.
6. $m\angle ABC > m\angle BAD$.	6. The measure of an obtuse angle is greater than the measure of an acute angle.
7. $\overline{AB} \cong \overline{AB}$.	7. Reflexive property of congruence.
8. $AC > BD$.	8. If two triangles have two sides of one congruent respectively to two sides of the other and the included angles are unequal, then the triangle which has the greater included angle has the greater third side.

EXERCISES

1. In triangle ABC, $AB = 5$ in., $BC = 10$ in., and $m\angle B = 40$. In triangle $A'B'C'$, $A'B' = 5$ in., $B'C' = 10$ in., and $m\angle B' = 90$. Prove that $A'C' > AC$.

2. $\overline{AC}$ and $\overline{BD}$ are diagonals of parallelogram $ABCD$. If $m\angle DAB = 50$, then (*a*) $AC = BD$ (*b*) $AC < BD$ (*c*) $AC > BD$.

3. If, in rhombus $ABCD$, $m\angle A = 110$, then (*a*) $AC = BD$ (*b*) $AC > BD$ (*c*) $AC < BD$.

4. In triangle DEF, $DE = 6$ in., $EF = 8$ in., and $FD = 10$ in. In triangle $D'E'F'$, $D'E' = 6$ in., $E'F' = 8$ in., and $D'F' = 12$ in.
a. Prove that $m\angle E' > m\angle E$.
b. Prove that $\angle E'$ is an obtuse angle.

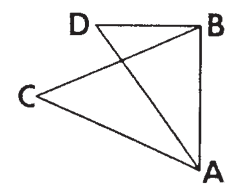
5. In parallelogram $ABCD$, with diagonals $\overline{AC}$ and $\overline{BD}$, $AC < BD$.
a. Prove that $m\angle CDA < m\angle DAB$.
b. Prove that $\angle CDA$ is an acute angle.

6. In triangle ABC, $\overline{CD}$ is a median to $\overline{AB}$. If $m\angle CDB > m\angle CDA$, prove that $CB > CA$.

7. In isosceles triangle RST, $\overline{RT} \cong \overline{TS}$. If Q is a point on $\overline{RS}$ such that $RQ > QS$, prove that $m\angle RTQ > m\angle QTS$.

8. If $\overline{AC} \cong \overline{AD}$, prove that $CB > DB$.

9. In triangle RST, $RS > ST$. P is a point on $\overline{RS}$ and Q is a point on $\overline{ST}$ such that $\overline{RP} \cong \overline{TQ}$. Prove that $RQ > TP$.



Ex. 8

6. Inequalities in Circles

Previously we have studied theorems about congruent (equal in measure) central angles, arcs, and chords in a circle or in equal circles. Now we will study theorems dealing with central angles, arcs, and chords which are of unequal measure when they are in the same circle or in equal circles.

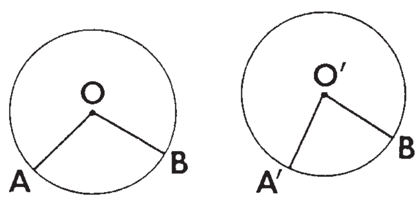


Fig. 9–16

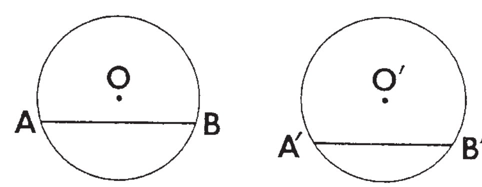


Fig. 9–17

Theorem 135. **In a circle or in equal circles, if two central angles are unequal, then their arcs are unequal, and the greater angle has the greater arc.**

In Fig. 9–16, if circle O = circle O' and the measure of central $\angle AOB >$ the measure of central $\angle A'O'B'$, then $\widehat{AB} > \widehat{A'B'}$.

Theorem 136. **In a circle or in equal circles, if two arcs are unequal, then their central angles are unequal, and the greater arc has the greater central angle.**

In Fig. 9–16, if circle O = circle O' and $\widehat{AB} > \widehat{A'B'}$, then the measure of central $\angle AOB >$ the measure of central $\angle A'O'B'$.

Theorem 137. **In a circle or in equal circles, if two chords are unequal, then their minor arcs are unequal, and the greater chord has the greater minor arc.**

In Fig. 9–17, if circle O = circle O' and the length of chord $\overline{AB} >$ the length of chord $\overline{A'B'}$, then $\widehat{AB} > \widehat{A'B'}$.

Theorem 138. **In a circle or in equal circles, if two minor arcs are unequal, then their chords are unequal, and the greater minor arc has the greater chord.**

In Fig. 9–17, if circle O = circle O' and $\widehat{AB} > \widehat{A'B'}$, then the length of chord $\overline{AB} >$ the length of chord $\overline{A'B'}$.

***Theorem 139.* In a circle or in equal circles, if two chords are unequal, then they are at unequal distances from the center, and the greater chord is the smaller distance from the center.**

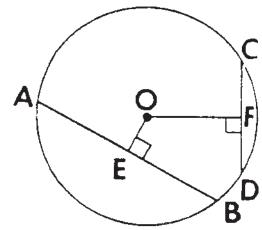


Fig. 9–18

In circle O (Fig. 9–18), if $AB > CD$, $\overline{OE} \perp \overline{AB}$, and $\overline{OF} \perp \overline{CD}$, then $OE < OF$.

***Theorem 140.* In a circle or in equal circles, if two chords are not equally distant from the center, then they are unequal, and the chord that is the smaller distance from the center is the greater chord.**

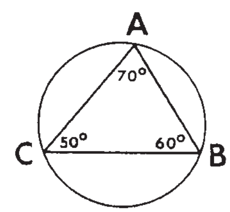
In circle O (Fig. 9–18), if $\overline{OE} \perp \overline{AB}$, $\overline{OF} \perp \overline{CD}$, and $OE < OF$, then $AB > CD$.

Methods of Proof:

1. To prove that two arcs are unequal, show that they are in a circle or in equal circles and that:
   a. their central angles are unequal. OR
   b. their chords are unequal.
2. To prove that two chords are unequal, show that they are in a circle or in equal circles and that:
   a. their arcs are unequal. OR
   b. they are not equally distant from the center.

MODEL PROBLEM

Triangle ABC is inscribed in a circle. If $m\angle A = 70$, $m\angle B = 60$, and $m\angle C = 50$, which side of triangle ABC is nearest the center of the circle?



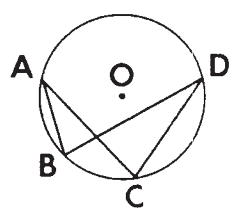
Solution:

1. $\angle A$, $\angle B$, and $\angle C$ are inscribed angles.
2. Since $m\angle A = 70$, $m\overset{\frown}{CB} = 140$.
3. Since $m\angle B = 60$, $m\overset{\frown}{AC} = 120$.
4. Since $m\angle C = 50$, $m\overset{\frown}{AB} = 100$.
5. Since $m\overset{\frown}{CB} > m\overset{\frown}{AC} > m\overset{\frown}{AB}$, then $\overset{\frown}{CB} > \overset{\frown}{AC} > \overset{\frown}{AB}$.
6. Since $\overset{\frown}{CB} > \overset{\frown}{AC} > \overset{\frown}{AB}$, then $CB > AC > AB$ because in a circle the greatest of several unequal arcs has the greatest chord.
7. Since chord $\overline{CB}$ is the greatest of the unequal chords, it is nearest the center of the circle because in a circle the greatest of several chords is nearest the center.

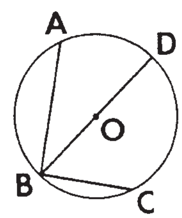
Answer: Side $\overline{CB}$.

EXERCISES

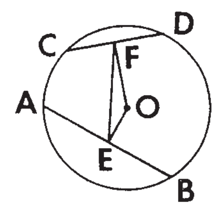
1. Triangle ABC is inscribed in a circle. If $m\angle A = 100$, $m\angle B = 50$, and $m\angle C = 30$ (*a*) name the largest intercepted arc (*b*) name the side of the triangle that is nearest the center of the circle.
2. Triangle RST is inscribed in a circle. If $m\angle R = 85$ and $m\angle S = 45$ (*a*) name the smallest intercepted arc (*b*) name the side of the triangle which is farthest from the center of the circle.
3. If acute triangle ABC is inscribed in a circle and $m\angle B > m\angle A$, then $\overset{\frown}{BC}$ is (*a*) greater than $\overset{\frown}{AC}$ (*b*) equal to $\overset{\frown}{AC}$ (*c*) less than $\overset{\frown}{AC}$.
4. Triangle ABC is inscribed in a circle. $m\angle A = 80$, $m\angle B = 70$, and $m\angle C = 30$. Which side of triangle ABC is nearest the center of the circle?
5. Triangle DEF is inscribed in a circle. If $m\angle E > m\angle D$, prove that $\overset{\frown}{DF} > \overset{\frown}{EF}$.



Ex. 6



Ex. 7



Ex. 8

6. In circle O, $BD > AC$. Prove that $CD > AB$.
7. In circle O, $\overline{BD}$ is a diameter, $\overline{AB}$ and $\overline{BC}$ are chords, and $AB > BC$. Prove that $m\angle ABD < m\angle CBD$.
8. In circle O, $\overline{OE} \perp \overline{AB}$, $\overline{OF} \perp \overline{CD}$, and $m\angle OEF > m\angle OFE$. Prove that $AB > CD$.
9. An equilateral triangle and a regular hexagon are inscribed in the same circle. Prove that the length of an apothem of the hexagon is greater than the length of an apothem of the equilateral triangle.
10. Triangle ABC is inscribed in circle O and $m\angle A > m\angle B$. Prove that $\overline{BC}$ is nearer the center of the circle than $\overline{AC}$ is.
11. $\overline{AB}$ is a chord of a circle and C is the midpoint of minor arc $\overset{\frown}{AB}$. Chord $\overline{CD}$ is drawn intersecting $\overline{AB}$ at E. Chords $\overline{BD}$ and $\overline{BC}$ are drawn. Prove that if BE is greater than EC, then BD is greater than BC.
12. *Prove:* The shortest chord that can be drawn through a point inside a circle is the chord which is perpendicular to the radius drawn through this point.