CHAPTER II

Methods of Arriving at Conclusions

1. Using Observation to Arrive at Conclusions

In our daily lives, there are many situations in which we find it necessary to investigate the truth or falsity of a statement. A statement that is either true or false we will call a *proposition.* Frequently, *direct observation* is the most efficient method of investigation to use in determining the truth or falsity of a proposition. For example, the truth or falsity of each of the following statements can be quickly and correctly determined by observation:

1. It is snowing.
2. The radio is playing.
3. There are 35 students in the room.

Scientists commonly employ the method of observation in their work. However, one may draw wrong conclusions from observation alone. For example, see how your eyes will deceive you when observing the following figures:

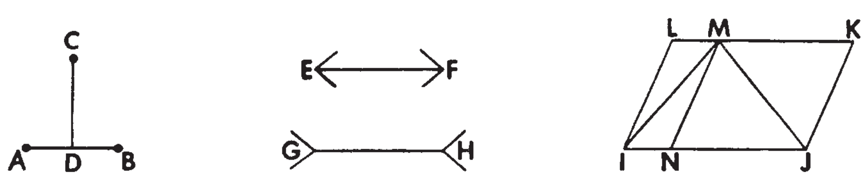


Fig. 2–1 **Fig. 2–2** **Fig. 2–3**

1. In Fig. 2–1, does the length of $\overline{AB}$ appear to be equal to the length of $\overline{CD}$? Check your answer by measurement.
2. In Fig. 2–2, does the length of $\overline{GH}$ appear to be greater than the length of $\overline{EF}$? Check your answer by measurement.
3. In Fig. 2–3, does the length of $\overline{IM}$ appear to be less than the length of $\overline{MJ}$? Check your answer by measurement.

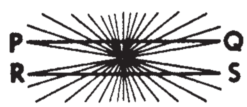


Fig. 2–4

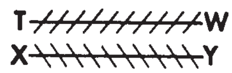


Fig. 2–5

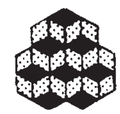


Fig. 2–6

4. In Fig. 2–4, does the figure named by P and Q, also the figure named by R and S, appear to be curved or straight? Check your answer with a straightedge.
5. In Fig. 2–5, is the distance from T to X the same as the distance from W to Y? Check your answer by measurement.
6. In Fig. 2–6, are there 6 or 7 cubes?

Observation alone will help us arrive at conclusions in some situations. In other cases, additional methods should be used, such as reasoning or experimentation which uses measurement.

2. Using Experimentation and Measurement to Arrive at Conclusions—Inductive Reasoning

In order to determine the results that must follow from a given set of conditions, scientists often perform experiments in which measurements are made. We must realize that their results are approximate, because it is impossible to make exact measurements. *All direct measurements are approximate.*

In geometry, too, we can perform experiments in which measurements play an important role. These experiments may help us to discover properties of geometric figures and to determine geometric relationships.

If we look at an isosceles triangle, $\triangle ABC$, in which the vertex angle, $\angle A$, is acute, and $\overline{AB} \cong \overline{AC}$, we may observe that the base angles, $\angle B$ and $\angle C$, appear to be congruent (see Fig. 2–7). In order to check our observation, we can measure $\angle B$ and $\angle C$ with a protractor. When we do, we discover that $\angle B$ and $\angle C$ have the same measure. Therefore, $\angle B \cong \angle C$.

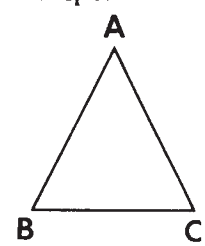


Fig. 2–7

To convince ourselves that this relationship is also true in other isosceles triangles, we can draw another isosceles triangle and then measure its base angles. In $\triangle ABC$ (see Fig. 2–8), the vertex angle, $\angle A$, is a right angle. We can measure $\angle B$ and $\angle C$ with a protractor, and once more we would discover that the base angles have the same measure and are therefore congruent.

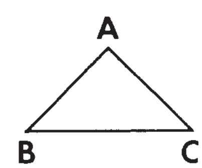


Fig. 2–8

To convince ourselves still further, we can draw an isosceles triangle, $\triangle ABC$, whose vertex angle, $\angle A$, is an obtuse angle (see Fig. 2–9). In this triangle, too, we can measure the base angles, $\angle B$ and $\angle C$, and we would discover once again that they have the same measure and are congruent.

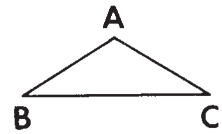


Fig. 2–9

We may repeat the experiment several more times, changing the number of degrees in the vertex angle and changing the lengths of the sides of the isosceles triangles. In each experiment, we will find that the measure of one base angle, $\angle B$, is the same as the measure of the other base angle, $\angle C$.

From this group of experiments, dealing with various types of isosceles triangles, we are ready to arrive at the general conclusion that "the base angles of an isosceles triangle are congruent." This method of reasoning, which arrives at a general truth by studying a series of particular examples, is called *inductive reasoning*.

To discover a general geometric relationship by experimenting with particular examples, that is, to engage in inductive reasoning, we proceed as follows:

1. Carefully state the problem.
2. Perform a number of varied experiments.
3. Make the necessary measurements and observations.
4. Study and compare the results (the data).
5. Draw a proper general conclusion.

When we use inductive reasoning, we must exercise extreme care. Since our data may be based on measurement and observation which may not be absolutely accurate, our conclusion may not be accurate. Also, in inductive reasoning, we arrive at a general conclusion before we have examined every possible example. A single counterexample, that is, an example for which the general conclusion arrived at by inductive reasoning is false, is sufficient to show that the general conclusion is false. For example, after examining several isosceles triangles, Henry made the generalization that "all isosceles triangles are acute triangles." When his teacher showed him an isosceles triangle one of whose angles was a right angle, Henry realized that his generalization was false. When a general conclusion is reached by inductive reasoning alone, it can at best be called "probably true." Furthermore, although inductive reasoning may be a powerful aid in discovering new facts, it does not help us in explaining or proving them.

EXERCISES

1. *a.* Draw three right triangles that have different sizes and shapes.

If one accepts statements 1 and 2 as true, then one must accept as true the conclusion:

3. I shall die someday.

This is an example of valid reasoning.

If we know that an angle which contains 90° is a right angle, and that angle A contains 90°, we can conclude that angle A is a right angle by reasoning as follows:

1. All angles that contain 90° are right angles.
2. Angle A contains 90°.
3. Angle A is a right angle.

This is another example of valid reasoning.

Notice the three kinds of statements that are found in each of the preceding examples of a three-step reasoning process:

Step 1 is a general statement concerning a whole group. This statement is called the *major premise.*

Step 2 is a specific statement which indicates that an individual is a member of that group. This statement is called the *minor premise.*

Step 3 is a statement to the effect that the general statement which applies to the group also applies to the individual. This statement is called the *deduced statement*, or the *deduction.*

Such a chain of reasoning is called a *syllogism.* When we use syllogisms to arrive at conclusions, we are engaging in *deductive reasoning.* In deductive reasoning, we arrive at conclusions from accepted premises. If both the major premise and the minor premise are true, then the conclusion must be true, and the reasoning is valid.

However, unless we are very careful when using this type of reasoning, we can easily make errors in reasoning.

For example, since all seats in the orchestra at the performance of a certain play cost $40.00, are we justified in concluding that Mr. Somers, who is sitting in the orchestra, paid $40.00 for his seat? Let us see the chain of reasoning.

1. All people who buy tickets for orchestra seats for this show pay $40.00 for a ticket.
2. Mr. Somers is occupying an orchestra seat at this show.
3. Mr. Somers paid $40.00 for his orchestra seat.

Notice that in step 2, Mr. Somers failed to meet the general requirement set down for the whole group of people who bought orchestra seats. There is no statement to the effect that Mr. Somers bought a ticket. A friend might have given him a ticket. Hence, we cannot conclude that Mr. Somers paid $40.00 for his orchestra seat. If we did arrive at such a conclusion, our reasoning would be invalid.

When we engage in deductive reasoning using a syllogism:

KEEP IN MIND

> If a major premise which is true is followed by an appropriate minor premise which is true, a conclusion can be deduced which must be true, and the reasoning is valid.
>
> BUT
>
> If a major premise which is true is followed by an inappropriate minor premise which is true, a conclusion cannot be deduced.

MODEL PROBLEMS

1. How may the following three statements be arranged so that the first two will make it possible to deduce the third? (*a*) An eagle has feathers. (*b*) All birds have feathers. (*c*) An eagle is a bird.

Solution: If we arrange the sentences as a syllogism, we will have answered the question.

Step 1 should be the major premise.	(*b*) All birds have feathers.
Step 2 should be the minor premise.	(*c*) An eagle is a bird.
Step 3 should be the conclusion.	(*a*) An eagle has feathers.

Answer: (*b*), (*c*), (*a*).

2. All residents of this state who are registered voters are over 18 years of age. If John is a resident of this state, by valid reasoning we can conclude that: (*a*) If John is over 18, he is a registered voter. (*b*) If John is a registered voter, he is over 18. (*c*) If John is not a registered voter, he is not over 18.

Solution: The given statements and (*b*) can form a syllogism.

1. Major premise:	All residents of this state who are registered voters are over 18 years of age.
2. Minor premise:	John, a resident of this state, is a registered voter.
3. Conclusion:	John is over 18.

Answer: (*b*).

EXERCISES

In 1–5, deduce a conclusion using a syllogism.

1. *a.* All fish swim.
b. A trout is a fish.

2. *a.* All living human beings breathe.
b. Jim Chao is a living human being.

3. *a.* In a certain town, teachers must be at least 21 years old.
b. Miss Welsh teaches in that town.

4. *a.* In a right triangle, one angle is a right angle.
b. Triangle ABC is a right triangle.

5. *a.* In a circle, all radii are congruent.
b. In circle O, segment $\overline{OA}$ and segment $\overline{OB}$ are radii.

In 6–9, write the syllogism which justifies the conclusion.

6. Since all straight angles contain 180°, angle A, which is a straight angle, contains 180°.

7. Since the sides of an equilateral triangle are equal in length, in equilateral triangle ABC, $AB = BC = AC$.

8. Since all licensed drivers in a certain state must be over 18 years of age, Jack Welsh, who is a licensed driver in that state, is over 18 years of age.

9. Since all Presidents of the United States must be native-born citizens, President George W. Bush was born in the United States.

In 10–13, arrange the three statements in the proper order so that the first two will make it possible to deduce the third.

10. *a.* Sally Aguello is a student in Evans High School.
b. All pupils in Evans High School are between 13 and 19 years of age.
c. Sally Aguello is between 13 and 19 years of age.

11. *a.* Sam bought his newspaper at Mr. Wilson's stand.
b. Sam always buys his newspaper at Mr. Wilson's stand.
c. Sam bought a newspaper.

12. *a.* A triangle that has two congruent sides is an isosceles triangle.
b. Triangle ABC is an isosceles triangle.
c. Triangle ABC has two congruent sides.

13. *a.* Line $\overleftrightarrow{AB}$ bisects the given line segment.
b. A line which passes through the midpoint of a given line segment bisects the given line segment.
c. Line $\overleftrightarrow{AB}$ passes through the midpoint of the given line segment.

In 14–17, state whether the reasoning which is used to deduce the conclusion is valid. If the reasoning is not valid, state the error in reasoning.

14. Every student in West High School is a member of the Student Organization. Sue Tucker, who is a student in West High School, is a member of the Student Organization.

15. Since all children in Barter City between the ages of 6 and 17 must attend school, Harry, who is 16 years old, attends school.

16. In all isosceles triangles, there are two congruent sides. Therefore, in triangle ABC, there are two congruent sides.

17. All quadrilaterals with four right angles are rectangles. Therefore, a polygon in which angles A, B, C, and D are right angles is a rectangle.

18. All children of Cedar City who are pupils in East High School are over 12 years of age. If Ted is a resident of Cedar City, by valid reasoning we can conclude that: (a) If Ted is not a pupil in East High School, he is not over 12 years of age. (b) If Ted is over 12 years of age, he is a pupil in East High School. (c) If Ted is a pupil in East High School, he is over 12 years of age.

4. Determining the Hypothesis and Conclusion of a Statement

The Hypothesis and Conclusion in an "If-Then" Sentence

A statement which is in the form "If ____, then ____" is called a *conditional statement.*

An example of a conditional statement is: *If two sides of a triangle are congruent, then the triangle is an isosceles triangle.*

This conditional statement is a complex sentence which contains two clauses, a *dependent clause* and an *independent clause.*

1. The dependent clause is introduced by the word *if.*

If two sides of a triangle are congruent

The part of this clause which follows the word *if* is called the *hypothesis,* or the *given.*

If we represent "Two sides of a triangle are congruent" (the statement in the dependent clause which follows the word *if*) by p, then p is the hypothesis, or given.

2. The independent clause is introduced by the word *then.*

then the triangle is an isosceles triangle

The part of this clause which follows the word *then* is called the *conclusion,* or the *to prove.*

If we represent "The triangle is an isosceles triangle" (the statement in the independent clause which follows the word *then*) by q, then q is the conclusion, or to prove.

In general, in a conditional statement which is written in the form "If p, then q," the hypothesis, or given, is p, and the conclusion, or to prove, is q.

Sometimes the if-clause appears at the end of the sentence. For example, the preceding "if-then" sentence may be restated as follows: *A triangle is an isosceles triangle if two sides of the triangle are congruent.*

The Hypothesis and Conclusion in a Simple Sentence

Consider the following statement:

Right angles A and B are congruent.

This simple sentence has a subject and a predicate.

1. The subject, Right angles A and B, is the hypothesis.
2. The predicate, are congruent, is the conclusion.

In general, if a statement is written as a simple sentence, the subject of the sentence is the hypothesis and the predicate is the conclusion.

In order to help us determine the hypothesis and conclusion in a simple sentence, we can rewrite the sentence as a conditional statement. We can do this by writing the word *If* before the subject of the sentence and the word *then* before the predicate of the sentence. For example, the preceding simple sentence (Right angles A and B are congruent.) may be restated as follows: *If angles A and B are right angles, then they are congruent.*

MODEL PROBLEMS

1. Identify the hypothesis and conclusion in the statement "If Mary has a temperature, then she is probably sick."

Solution: If Mary has a temperature, then she is probably sick.

The if-clause, "Mary has a temperature," is the hypothesis.
The then-clause, "she is probably sick," is the conclusion.

2. Identify the hypothesis and conclusion in the statement "The median drawn to the base of an isosceles triangle is perpendicular to the base."

Solution: The median drawn to the base of an isosceles triangle is perpendicular to the base.

The subject of the sentence, "The median drawn to the base of an isosceles triangle," is the hypothesis.

The predicate of the sentence, "is perpendicular to the base," is the conclusion.

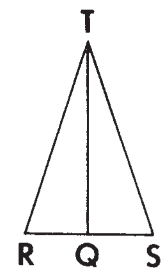
3. For the following statement, draw a figure, letter it, and state in terms of the letters of the figure what is the hypothesis and what is the conclusion:

If the bisector of the vertex angle of an isosceles triangle is drawn, the bisector is perpendicular to the base of the triangle.

Solution:

The if-clause, "the bisector of the vertex angle of an isosceles triangle is drawn," is the hypothesis.

The then-clause, "the bisector is perpendicular to the base of the triangle," is the conclusion.



Answer: Hypothesis: Isosceles triangle RST with $\overline{TR} \cong \overline{TS}$. $\overline{TQ}$ bisects $\angle RTS$.
Conclusion: $\overline{TQ} \perp \overline{RS}$.

EXERCISES

In 1–13, state the hypothesis and the conclusion.

1. If a boy has a pleasant personality, he will make friends readily.
2. If metal is cooled, then it contracts.
3. If the sun is shining, it is a clear day.
4. If the sum of the measures of two angles is 90, the angles are complementary.
5. If two angles are congruent, their measures are equal.
6. Children are polite if they have good manners.
7. A triangle is equiangular if the triangle has three congruent sides.
8. Two angles are supplementary if they are right angles.
9. Clothes which are expensive are made of better materials.
10. A well-fed child gains weight.
11. Two lines that intersect form vertical angles.
12. The bisector of an angle divides the angle into two congruent angles.

13. The median to the base of an isosceles triangle bisects the vertex angle of the triangle.

In 14–18, draw a figure, letter it, and state in terms of the letters of the figure what is the hypothesis and what is the conclusion.

14. If the altitude is drawn to the base of an isosceles triangle, the altitude bisects the base.

15. If two angles of a triangle are congruent, the sides opposite these angles are congruent.

16. A triangle is isosceles if the bisector of an angle bisects the opposite side.

17. Altitudes drawn to the congruent sides of an isosceles triangle are congruent.

18. If the opposite sides of a quadrilateral are congruent, the diagonals of the quadrilateral bisect each other.

5. Deductive Reasoning Involving a Conditional Statement

Consider the following two statements:

1. If you hit a home run, then you will score a run.
2. You hit a home run.

If we accept statements 1 and 2 as true, then we must accept as true the conclusion:

3. You scored a run.

Notice that three kinds of statements are found in the previous example.

Statement 1 is a general conditional statement in the "If _____, then _____," form in which the if-clause is the hypothesis and the then-clause is the conclusion.

Statement 2 is a statement which indicates that you satisfied the conditions of the hypothesis.

Statement 3 is a statement, a deduction, which indicates that the conclusion is true for you.

In general, if p represents the hypothesis in a conditional statement and q represents the conclusion in this conditional statement, then we can perform deductive reasoning in the following manner:

If (1) the conditional statement "If p, then q" is true, and (2) the hypothesis p is true, then (3) the conclusion q is true.

Let us see how we can perform such deductive reasoning in a geometric example. (See Fig. 2–10.)

If (1) the conditional statement "If $\angle ABC$ contains between 0° and 90°, then $\angle ABC$ is an acute angle" is accepted as true and (2) the hypothesis "$\angle ABC$ contains between 0° and 90°" is asserted to be true, then (3) the conclusion "$\angle ABC$ is an acute angle" is deduced to be true.

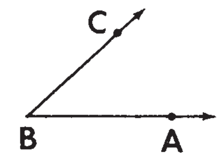


Fig. 2–10

Now let us see whether accepting the truth of a conditional statement and also the truth of the conclusion of the statement permits us to deduce that the hypothesis of the statement must be true. For example,

If (1) the conditional statement "If two angles are right angles, then they have equal measures" is accepted as true, and (2) the conclusion "Two angles have equal measures" is asserted to be true, does it follow that the hypothesis "The two angles are right angles" must be true? *Of course not!* If each of the angles contains 30°, the angles would have equal measures, yet they would not be right angles.

In general, if p represents the hypothesis in a conditional statement and q represents the conclusion in this conditional statement, then:

If (1) the conditional statement "If p, then q" is true, and (2) the conclusion q is true, *we cannot deduce that the hypothesis p is true. The hypothesis p may or may not be true.*

EXERCISES

In 1–4, state whether accepting the truth of statements (a) and (b) makes it possible to deduce the truth of statement (c). Give a reason for your answer.

1. *a.* If it is snowing, I am at home.
b. It is snowing.
c. Therefore, I am at home.

2. *a.* You are a resident of the United States if you are a resident of New York State.
b. You are a resident of the United States.
c. Therefore, you are a resident of New York State.

3. *a.* If two angles are straight angles, they have equal measures.
b. Two angles A and B are straight angles.
c. Therefore, the two angles A and B have equal measures.

4. *a.* If two angles are adjacent angles, then they have a common vertex.
b. Two angles ABC and ABD have a common vertex.
c. Therefore, the two angles ABC and ABD are adjacent angles.

In 5–8, assume the truth of statements (a) and (b). If from the truth of these statements it is possible to deduce the truth of a third statement, write the deduced statement and tell why the deduced statement is true. If from the truth of the first two statements it is not possible to deduce the truth of a third statement, tell why this is so.

5. *a.* If you attend Highview High School, you are a male student.
b. You attend Highview High School.

6. *a.* If you are a pretty girl, then you have many friends.
b. You have many friends.

7. *a.* If two angles are supplementary, then the sum of the measures of the two angles is 180.
b. Two angles *R* and *S* are supplementary.

8. *a.* Two angles are complementary if each of the two angles contains 45°.
b. Two angles *A* and *B* are complementary.

6. Understanding the Nature of a Postulational System

In order to avoid any misunderstanding in a discussion, it is essential that words and phrases be defined so that they will have the same meaning for all people.

For example, we can define an isosceles triangle as a triangle which has two congruent sides.

Now the question arises: What is a triangle? We can define a triangle as a polygon which has three sides.

What is a polygon? It is a closed figure in a plane which is the union of line segments.

What is a line segment? It is a set of points of a line consisting of two points on the line and the set of all points on the line between these two points.

What is a plane and what is a line? Now we find that we have no simpler, previously defined terms to use.

In geometry, therefore, we must have some undefined terms with which to begin the process of defining new terms.

In our study of the use of reasoning in arriving at conclusions, we saw that, to establish the truth or falsity of a proposition, we must offer other previously accepted propositions as convincing evidence. For example, we might wish to establish the truth of the following proposition: "If two line segments are congruent to the same line segment, they are congruent to each other." To do this, we might argue as follows (see Fig. 2–11):

1. $\overline{AB} \cong \overline{EF}$.
2. $\overline{CD} \cong \overline{EF}$.

Fig. 2–11

It appears that we should now be ready to grant that $\overline{AB} \cong \overline{CD}$. Yet if we were asked to state a previously proved proposition which would justify this conclusion, we would find none.

Thus, in geometry, we must start with some propositions whose truth we assume in order to begin the process of deducing the truth of other propositions.

A proposition whose truth is assumed is called an *assumption*, a *postulate*, or an *axiom*.

Some mathematicians reserve the term *axiom* for a general statement whose truth is assumed without proof, and the term *postulate* for a geometric statement whose truth is assumed without proof. We will use the term *postulate* for both types of assumptions.

Definition. **A *postulate* is a statement whose truth is accepted without proof.**

In a postulational system, the *undefined terms*, the *defined terms*, and the *postulates* are the seeds from which the tree of knowledge in the subject grows. They, together with the laws of reasoning, become the instruments with which we deduce, that is, prove the truth of, new statements, called *theorems*.

Definition. **A *theorem* is a statement proved by deduction.**

Postulational thinking, which is the instrument used to make deductions in demonstrative geometry, is a powerful technique for several reasons:

1. It makes it possible to arrive at a conclusion in situations where observation and measurement are not practical or possible.
2. Observation and measurement may help to discover a fact, but they never explain the reason for the truth of the fact. Postulational thinking accounts for, and explains why, the reasoning used in arriving at a conclusion is valid or invalid.
3. In a postulational system, a combination of the undefined terms, the defined terms, the postulates, and the derived theorems leads to the discovery and explanation of new theorems. In this way, the entire body of knowledge known as geometry is developed.

EXERCISES

1. Define a postulate.
2. Define a theorem.
3. Answer *true* or *false*. In a postulational system, all terms are defined.
4. What are the elements in a postulational system that may be used to explain a new theorem?

7. Using Postulates in Proving Conclusions

We have seen that deductive reasoning is based upon the use of undefined terms, defined terms, and postulates. Now we are going to examine some of these postulates and learn how to use them in deductive reasoning.

Equality Postulates

When we state the relation "a is equal to b," symbolized by "$a = b$," we mean that the symbol a and the symbol b are two different names for the same element of a set. Very frequently a and b represent a number. For example,

1. When we write $RS = LM$, we mean that line segment $\overline{RS}$ and line segment $\overline{LM}$ have the same length.
2. When we write $m\angle A = m\angle B$, we mean that angle A and angle B contain the same number of degrees.

We should now be ready to accept the following three equality postulates, which are also referred to as the *properties of equality*.

Postulate 1. The Reflexive Property of Equality

$$a = a$$

This property, which states that if there is an element of a set named a, then there is only one element of that set which is named by a, is sometimes called the *principle of identity*. We can restate the reflexive property of equality in words as follows:

Postulate 1. **A quantity is equal to itself.**

EXAMPLE 1 (see Fig. 2–12). $LM = LM$.

Fig. 2–12

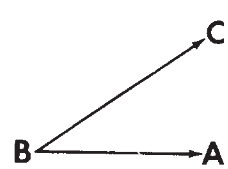


Fig. 2–13

EXAMPLE 2 (see Fig. 2–13). $m\angle ABC = m\angle ABC$.

Postulate 2. The Symmetric Property of Equality

If $a = b$, then $b = a$.

This property states that if a names the same element of a set as b, then b names the same element as a. We can restate the symmetric property of equality in words as follows:

***Postulate 2.* An equality may be reversed.**

EXAMPLE 1 (see Fig. 2–14). If $AB = CD$, then $CD = AB$.

EXAMPLE 2 (see Fig. 2–14). If $m\angle R = m\angle S$, then $m\angle S = m\angle R$.

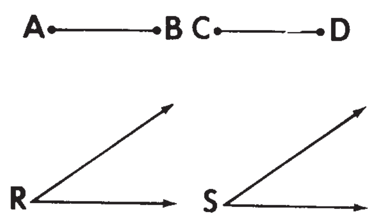


Fig. 2–14

Postulate 3. The Transitive Property of Equality

If $a = b$, and $b = c$, then $a = c$.

This property states that if a and b name the same element of a set, and b and c name the same element of that set, then a and c name the same element of the set. We can restate the transitive property of equality in words as follows:

***Postulate 3.* If quantities are equal to the same quantity, they are equal to each other.**

Now we will see how the symmetric and the transitive postulates of equality can be used in deductive reasoning.

EXAMPLE 1. If we know that $m\angle x = 40$ and $m\angle y = 40$, and we wish to prove that $m\angle x = m\angle y$, we can say that since $m\angle y = 40$, then $40 = m\angle y$, using the symmetric property of equality; also, we can say that, since $m\angle x = 40$, and $40 = m\angle y$, then $m\angle x = m\angle y$, using the transitive property of equality.

It is possible to arrange the preceding proof more formally in two columns. In order to arrange a proof in a formal manner, we will:

1. State the hypothesis, which is also called the *given*, because the hypothesis contains the given facts.
2. State the conclusion, which is also called the *to prove*, because the conclusion contains what is to be proved.

Note. In our work, we will use the terms *given* and *to prove* rather than the terms *hypothesis* and *conclusion*.

3. Present the *proof*, the deductive reasoning, which is the series of logical arguments used in the demonstration. Each step in the proof should consist of a *statement* in one column and its *reason* in the other column. A reason may be the *given*, a *definition*, a *postulate*, or, as we shall see later, a *previously proved theorem*.

Example 1 (see Fig. 2–15).

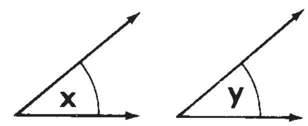


Fig. 2–15

Given: $m\angle x = 40$.
$m\angle y = 40$.

To prove: $m\angle x = m\angle y$.

Proof:

Statements	*Reasons*
1. $m\angle x = 40$.	1. Given.
2. $m\angle y = 40$.	2. Given.
3. $40 = m\angle y$.	3. The symmetric property of equality: An equality may be reversed.
4. $m\angle x = m\angle y$.	4. The transitive property of equality: If quantities are equal to the same quantity, they are equal to each other.

Note. In the future, we will abbreviate the proof by eliminating step 3 and reason 3, the step which makes use of the symmetric property of equality. After presenting steps 1 and 2, we will immediately deduce the conclusion that $m\angle x = m\angle y$.

Example 2 (see Fig. 2–16).

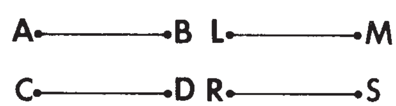


Fig. 2–16

Given: $AB = LM$.
$CD = RS$.
$LM = RS$.

To prove: $AB = CD$.

Proof:

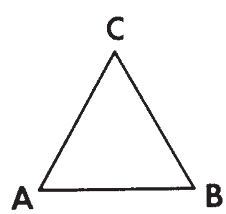
Statements	Reasons
1. $AB = LM$.	1. Given.
2. $LM = RS$.	2. Given.
3. $AB = RS$.	3. The transitive property of equality: If quantities are equal to the same quantity, they are equal to each other.
4. $CD = RS$.	4. Given.
5. $AB = CD$.	5. The transitive property of equality: If quantities are equal to the same quantity, they are equal to each other.

Notice that in example 2 we proved an illustration of the statement "If quantities are equal to equal quantities, they are equal to each other." In the future, we will feel free to use this statement, which is an expanded version of the transitive property of equality, as a reason in a proof.

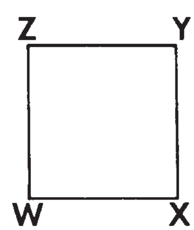
EXERCISES

In 1–3, (*a*) state the postulate that can be used to show that the conclusion is valid and (*b*) write a formal proof.

Ex. 1



Ex. 2



Ex. 3

1. *Given:* $CD = 2$ inches.
$XY = 2$ inches.

Prove: $CD = XY$.

2. *Given:* $m\angle A = m\angle B$.
$m\angle C = m\angle B$.

Prove: $m\angle A = m\angle C$.

3. *Given:* $WZ = XY$.
$ZY = WX$.
$WZ = ZY$.

Prove: $XY = WX$.

Equivalence Relations

Relations which have the reflexive property, the symmetric property, and the transitive property are called *equivalence relations*. We would say, therefore, that "is equal to (=)" for numbers is an equivalence relation because equality has all of the three required properties. On the other hand, the relation "is perpendicular to ($\perp$)" for lines is not an equivalence relation. Suppose a, b, and c represent lines. Consider the following statements:

1. Reflexive property (see Fig. 2–17): $a \perp a$.
2. Symmetric property (see Fig. 2–18): If $a \perp b$, then $b \perp a$.
3. Transitive property (see Fig. 2–19): If $a \perp b$, and $b \perp c$, then $a \perp c$.

Fig. 2–17

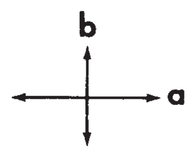


Fig. 2–18

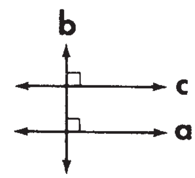


Fig. 2–19

Observe that the relation "is perpendicular to ($\perp$)" does have the symmetric property, but has neither the reflexive property nor the transitive property.

Congruence of Line Segments and Congruence of Angles

Since we have said that $\overline{AB} \cong \overline{CD}$ is equivalent to $AB = CD$, and $\overline{CD} \cong \overline{EF}$ is equivalent to $CD = EF$, it follows that the "congruence of line segments" has the same three properties as "the equality of numbers":

1. Reflexive property: $\overline{AB} \cong \overline{AB}$.
2. Symmetric property: If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.
3. Transitive property: If $\overline{AB} \cong \overline{CD}$, and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.

Hence, we say that "congruence of line segments" is an equivalence relation.

Since we have said that $\angle ABC \cong \angle DEF$ is equivalent to $m\angle ABC = m\angle DEF$, and $\angle DEF \cong \angle XYZ$ is equivalent to $m\angle DEF = m\angle XYZ$, it follows that the "congruence of angles" has the same three properties as "the equality of numbers":

1. Reflexive property: $\angle ABC \cong \angle ABC$.
2. Symmetric property: If $\angle ABC \cong \angle DEF$, then $\angle DEF \cong \angle ABC$.
3. Transitive property: If $\angle ABC \cong \angle DEF$, and $\angle DEF \cong \angle XYZ$, then $\angle ABC \cong \angle XYZ$.

Hence, we say that "congruence of angles" is an equivalence relation.

KEEP IN MIND

"Equality of numbers," "congruence of line segments," and "congruence of angles" are all equivalence relations. In the future, we will consider that the reflexive, the symmetric, and the transitive properties of "equality of numbers" are also to be the reflexive, the symmetric, and the transitive properties of "congruence of line segments" and of "congruence of angles."

EXERCISES

In 1–6, name the property that justifies the statement.

1. $\overline{AC} \cong \overline{AC}$.

2. If $\angle RST \cong \angle XYZ$, then $\angle XYZ \cong \angle RST$.

3. $\angle ABC \cong \angle ABC$.

4. If $\overline{LM} \cong \overline{XY}$, then $\overline{XY} \cong \overline{LM}$.

5. If $\overline{AB} \cong \overline{CD}$, and $\overline{CD} \cong \overline{ST}$, then $\overline{AB} \cong \overline{ST}$.

6. If $\angle XYZ \cong \angle RST$, and $\angle RST \cong \angle ABC$, then $\angle XYZ \cong \angle ABC$.

In 7–10, name the properties of an equivalence relation which are satisfied by the given relation.

7. "is greater than" (for natural numbers)

8. "is a factor of" (for natural numbers)

9. "is the father of" (for people)

10. "lives in the same house as" (for people)

Substitution Postulate

***Postulate 4.* A quantity may be substituted for its equal in any expression.**

EXAMPLE 1 (see Fig. 2–20).

Given: $AB = 2AD$.
$AD = DB$.

To prove: $AB = 2DB$.

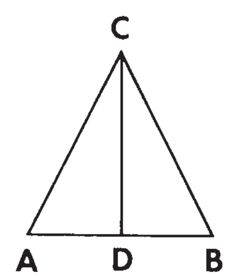


Fig. 2–20

[The proof is given on the next page.]

Proof: *Statements*	*Reasons*
1. $AB = 2AD$.	1. Given.
2. $AD = DB$.	2. Given.
3. $AB = 2DB$.	3. Substitution postulate: A quantity may be substituted for its equal in any expression.

EXAMPLE 2 (see Fig. 2–21).

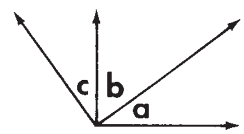


Fig. 2–21

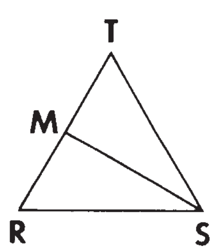
Given: $m\angle a + m\angle b = 90$.
$m\angle a = m\angle c$.

To prove: $m\angle c + m\angle b = 90$.

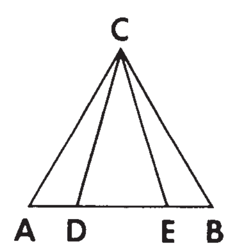
Proof: *Statements*	*Reasons*
1. $m\angle a + m\angle b = 90$.	1. Given.
2. $m\angle a = m\angle c$.	2. Given.
3. $m\angle c + m\angle b = 90$.	3. Substitution postulate: A quantity may be substituted for its equal in any expression.

EXERCISES

In 1–4, (*a*) state the postulate that can be used to show that the conclusion is valid and (*b*) write a formal proof.



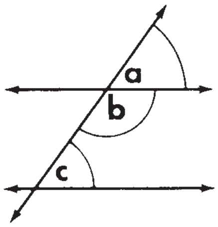
Ex. 1



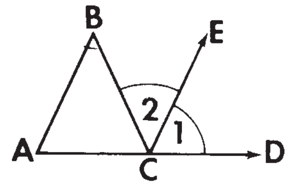
Ex. 2

1. *Given:* $MT = \frac{1}{2}RT$.
$RM = MT$.
Prove: $RM = \frac{1}{2}RT$.

2. *Given:* $AD + DE = AE$.
$AD = EB$.
Prove: $EB + DE = AE$.



Ex. 3



Ex. 4

3. *Given:* $m\angle a + m\angle b = 180$.
$m\angle a = m\angle c$.
Prove: $m\angle c + m\angle b = 180$.

4. *Given:* $m\angle 1 + m\angle 2 + m\angle BCA = 180$.
$m\angle A = m\angle 1$ and $m\angle B = m\angle 2$.
Prove: $m\angle A + m\angle B + m\angle BCA = 180$.

Partition Postulate

***Postulate 5.* A whole quantity is equal to the sum of all its parts.**

EXAMPLE 1 (see Fig. 2–22). $AD = AB + BC + CD$.

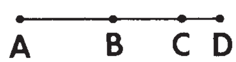


Fig. 2–22

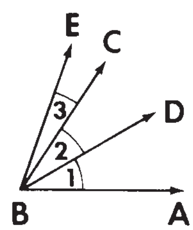


Fig. 2–23

EXAMPLE 2 (see Fig. 2–23). $m\angle ABE = m\angle 1 + m\angle 2 + m\angle 3$.

Addition Postulate

***Postulate 6.* If $a = b$, and $c = d$, then $a + c = b + d$.**

We can restate the addition postulate of equality as follows:

***Postulate 6.* If equal quantities are added to equal quantities, the sums are equal.**

EXAMPLE 1 (see Fig. 2–24).

Given: $AB = DE$.
$BC = EF$.

To prove: $AC = DF$.

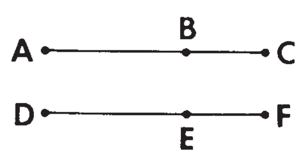


Fig. 2–24

[The proof is given on the next page.]

Proof: *Statements*	*Reasons*
1. $AB = DE$.	1. Given.
2. $BC = EF$.	2. Given.
3. $AB + BC = DE + EF$.	3. Addition postulate: If equal quantities are added to equal quantities, the sums are equal.
4. $AB + BC = AC$ and $DE + EF = DF$.	4. Partition postulate: A whole quantity is equal to the sum of all its parts.
5. $AC = DF$.	5. Substitution postulate: A quantity may be substituted for its equal in any expression.

We have learned that $\overline{AB} \cong \overline{DE}$ is equivalent to $AB = DE$, and that $\overline{BC} \cong \overline{EF}$ is equivalent to $BC = EF$. Hence, we will deal with the addition of congruent segments in the same way that we deal with the addition of segments whose lengths are equal. For example, when we are dealing with congruent segments, we will feel free to state the Addition Postulate as follows: "If congruent segments are added to congruent segments, the sums are congruent." For similar reasons, we will deal with the addition of congruent angles in the same way that we would deal with the addition of angles whose measures are equal. For example, when we are dealing with congruent angles, we will feel free to state the Addition Postulate as follows: "If congruent angles are added to congruent angles, the sums are congruent." See how this is done in the following examples.

EXAMPLE 2 (see Fig. 2–25).

Given: $\overline{AB} \cong \overline{DE}$.
$\overline{BC} \cong \overline{EF}$.

To prove: $\overline{AC} \cong \overline{DF}$.

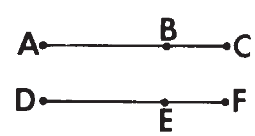


Fig. 2–25

Proof: *Statements*	*Reasons*
1. $\overline{AB} \cong \overline{DE}$.	1. Given.
2. $\overline{BC} \cong \overline{EF}$.	2. Given.
3. $\overline{AB} + \overline{BC} \cong \overline{DE} + \overline{EF}$, or $\overline{AC} \cong \overline{DF}$.	3. Addition postulate: If congruent segments are added to congruent segments, the sums are congruent segments.
NOTE. $\overline{AC}$ is another name for $\overline{AB} + \overline{BC}$; $\overline{DF}$ is another name for $\overline{DE} + \overline{EF}$.	

EXAMPLE 3 (see Fig. 2–26).

Given: $\angle ABG \cong \angle DEH$.
$\angle GBC \cong \angle HEF$.

To prove: $\angle ABC \cong \angle DEF$.

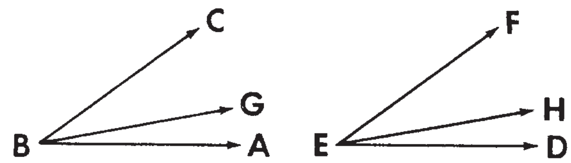


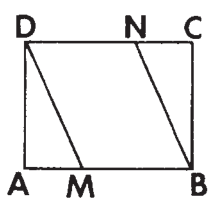
Fig. 2–26

Proof:

Statements	*Reasons*
1. $\angle ABG \cong \angle DEH$.	1. Given.
2. $\angle GBC \cong \angle HEF$.	2. Given.
3. $\angle ABG + \angle GBC \cong \angle DEH + \angle HEF$, or $\angle ABC \cong \angle DEF$.	3. Addition postulate: If congruent angles are added to congruent angles, the sums are congruent angles.
NOTE. $\angle ABC$ is another name for $\angle ABG + \angle GBC$; $\angle DEF$ is another name for $\angle DEH + \angle HEF$.	

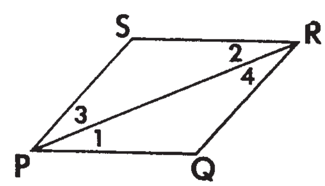
EXERCISES

In 1–6, (*a*) state the postulate or postulates that can be used to show that the conclusion is valid and (*b*) write a formal proof.



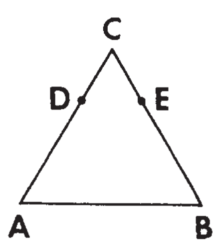
Ex. 1

1. *Given:* $AM = CN$.
$MB = ND$.
Prove: $AB = CD$.



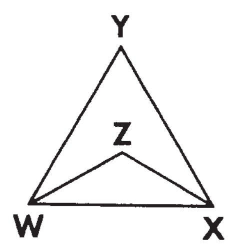
Ex. 2

2. *Given:* $m\angle 1 = m\angle 2$.
$m\angle 3 = m\angle 4$.
Prove: $m\angle QPS = m\angle QRS$.



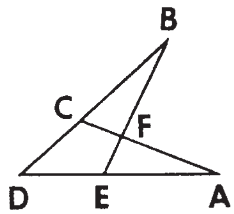
Ex. 3

3. *Given:* $\overline{CD} \cong \overline{CE}$.
$\overline{DA} \cong \overline{EB}$.
Prove: $\overline{CA} \cong \overline{CB}$.

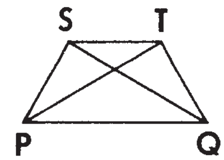


Ex. 4

4. *Given:* $\angle XWZ \cong \angle WXZ$.
$\angle ZWY \cong \angle ZXY$.
Prove: $\angle XWY \cong \angle WXY$.



Ex. 5



Ex. 6

5. *Given:* $\overline{AF} \cong \overline{BF}$.
$\overline{FC} \cong \overline{FE}$.
Prove: $\overline{AC} \cong \overline{BE}$.

6. *Given:* $\angle QST \cong \angle PTS$.
$\angle QSP \cong \angle PTQ$.
Prove: $\angle PST \cong \angle QTS$.

Subtraction Postulate

Postulate 7. **If $a = b$, and $c = d$, then $a - c = b - d$.**

We can restate the subtraction postulate of equality as follows:

Postulate 7. **If equal quantities are subtracted from equal quantities, the differences are equal.**

EXAMPLE 1 (see Fig. 2–27).

Given: $m\angle DAC = m\angle ECA$.
$m\angle 1 = m\angle 2$.

To prove: $m\angle 3 = m\angle 4$.

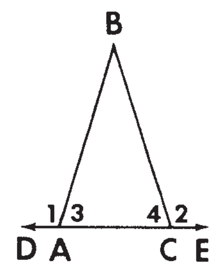


Fig. 2–27

Proof: *Statements*	*Reasons*
1. $m\angle DAC = m\angle ECA$.	1. Given.
2. $m\angle 1 = m\angle 2$.	2. Given.
3. $m\angle DAC - m\angle 1 = m\angle ECA - m\angle 2$, or $m\angle 3 = m\angle 4$.	3. Subtraction postulate: If equal quantities are subtracted from equal quantities, the differences are equal.

We will deal with the subtraction of congruent angles in the same way that we deal with the subtraction of angles which have equal measures. Also, we will deal with the subtraction of congruent segments in the same way that we deal with the subtraction of segments whose lengths are equal. See how this is illustrated in the following example:

EXAMPLE 2 (see Fig. 2–28).

Given: $\overline{AB} \cong \overline{AC}$.
$\overline{DB} \cong \overline{EC}$.

To prove: $\overline{AD} \cong \overline{AE}$.

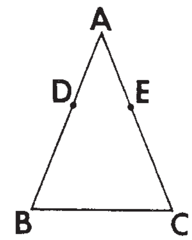
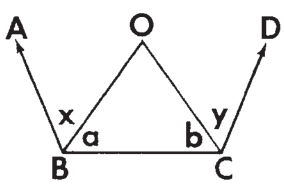


Fig. 2–28

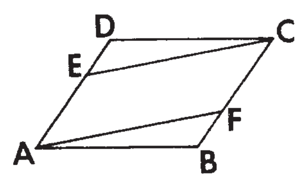
Proof: *Statements*	*Reasons*
1. $\overline{AB} \cong \overline{AC}$.	1. Given.
2. $\overline{DB} \cong \overline{EC}$.	2. Given.
3. $\overline{AB} - \overline{DB} \cong \overline{AC} - \overline{EC}$, or $\overline{AD} \cong \overline{AE}$.	3. Subtraction postulate: If congruent segments are subtracted from congruent segments, the differences are congruent segments.

EXERCISES

In 1–6, (*a*) state the postulate or postulates that can be used to show that the conclusion is valid and (*b*) write a formal proof.



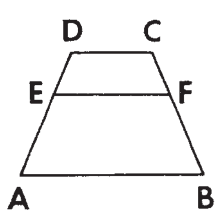
Ex. 1



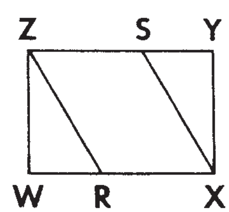
Ex. 2

1. *Given:* $m\angle ABC = m\angle DCB$.
$m\angle a = m\angle b$.
Prove: $m\angle x = m\angle y$.

2. *Given:* $AD = BC$.
$AE = CF$.
Prove: $DE = BF$.



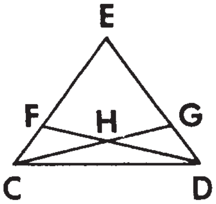
Ex. 3



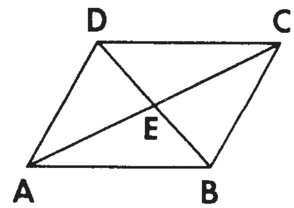
Ex. 4

3. *Given:* $\overline{DA} \cong \overline{CB}$.
$\overline{DE} \cong \overline{CF}$.
Prove: $\overline{EA} \cong \overline{FB}$.

4. *Given:* $\angle WZY \cong \angle WXY$.
$\angle RZY \cong \angle RXS$.
Prove: $\angle WZR \cong \angle YXS$.



Ex. 5



Ex. 6

5. *Given:* $\overline{FD} \cong \overline{GC}$.
$\overline{FH} \cong \overline{GH}$.
Prove: $\overline{CH} \cong \overline{DH}$.

6. *Given:* $\angle DEB \cong \angle AEC$.
$\angle DEA \cong \angle BEC$.
Prove: $\angle AEB \cong \angle DEC$.

Multiplication Postulate

***Postulate 8.* If $a = b$, and $c = d$, then $ac = bd$.**

We can restate the multiplication postulate of equality as follows:

***Postulate 8.* If equal quantities are multiplied by equal quantities, the products are equal.**

When each of two equal quantities is multiplied by the number 2, we have a special case of this postulate which is stated as follows:

Doubles of equal quantities are equal.

EXAMPLE 1 (see Fig. 2–29).

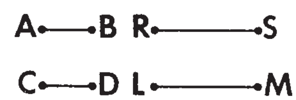


Fig. 2–29

Given: $AB = CD$.
$RS = 2AB$.
$LM = 2CD$.

To prove: $RS = LM$.

Proof:

Statements	*Reasons*
1. $AB = CD$.	1. Given.
2. $RS = 2AB$.	2. Given.
3. $LM = 2CD$.	3. Given.
4. $RS = LM$.	4. Multiplication postulate: Doubles of equal quantities are equal.

We will deal with doubles of congruent segments and doubles of congruent angles in the same way that we deal with doubles of segments whose lengths are equal, and doubles of angles which have equal measures. See how this is illustrated in the following example:

EXAMPLE 2 (see Fig. 2–30).

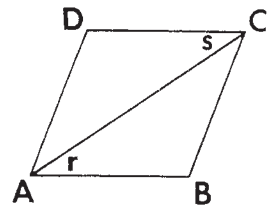


Fig. 2–30

Given: $\angle r \cong \angle s$.
$m\angle BAD = 2m\angle r$.
$m\angle BCD = 2m\angle s$.

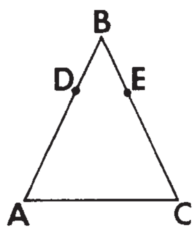
To prove: $\angle BAD \cong \angle BCD$.

Proof:

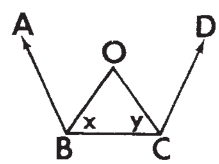
Statements	Reasons
1. $\angle r \cong \angle s$.	1. Given.
2. $m\angle BAD = 2m\angle r$.	2. Given.
3. $m\angle BCD = 2m\angle s$.	3. Given.
4. $\angle BAD \cong \angle BCD$.	4. Doubles of congruent angles are congruent.

EXERCISES

In 1–6, (*a*) state the postulate or postulates that can be used to show that the conclusion is valid and (*b*) write a formal proof.



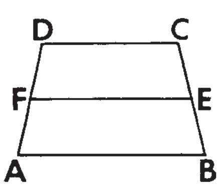
Ex. 1



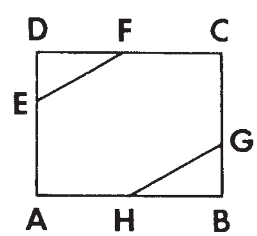
Ex. 2

1. *Given:* $BD = BE$.
$BA = 3BD$.
$BC = 3BE$.
Prove: $BA = BC$.

2. *Given:* $m\angle x = m\angle y$.
$m\angle CBA = 2m\angle x$.
$m\angle BCD = 2m\angle y$.
Prove: $m\angle CBA = m\angle BCD$.



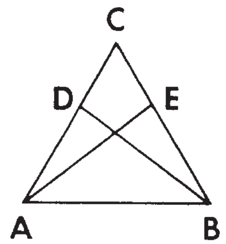
Ex. 3



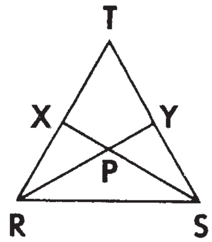
Ex. 4

3. *Given:* $AF = BE$.
$AD = 2AF$.
$BC = 2BE$.
Prove: $AD = BC$.

4. *Given:* $\overline{DF} \cong \overline{HB}$.
$DC = 2DF$.
$AB = 2HB$.
Prove: $\overline{DC} \cong \overline{AB}$.



Ex. 5



Ex. 6

5. *Given:* $\angle EBD \cong \angle EAD$.
$m\angle ABE = 2m\angle EBD$.
$m\angle BAD = 2m\angle EAD$.
Prove: $\angle ABE \cong \angle BAD$.

6. *Given:* $\overline{PX} \cong \overline{PY}$.
$SP = 2PX$.
$RP = 2PY$.
Prove: $\overline{SP} \cong \overline{RP}$.

Division Postulate

Postulate 9. **If $a = b$, and $c = d$, then $\frac{a}{c} = \frac{b}{d}$. ($c$ is not 0 and d is not 0.)**

We can restate the division postulate of equality as follows:

Postulate 9. **If equal quantities are divided by equal quantities (not zero), the quotients are equal.**

When each of two equal quantities is divided by the number 2, we have a special case of this postulate which is stated as follows:

Halves of equal quantities are equal.

EXAMPLE 1 (see Fig. 2–31).

Given: $m\angle LED = m\angle LFB$.
$m\angle 1 = \frac{1}{2}m\angle LED$.
$m\angle 2 = \frac{1}{2}m\angle LFB$.

To prove: $m\angle 1 = m\angle 2$.

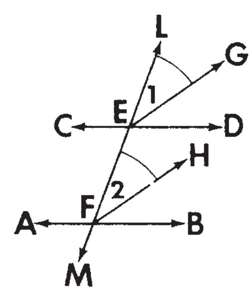


Fig. 2–31

Proof: *Statements*	*Reasons*
1. $m\angle LED = m\angle LFB$.	1. Given.
2. $m\angle 1 = \frac{1}{2}m\angle LED$.	2. Given.
3. $m\angle 2 = \frac{1}{2}m\angle LFB$.	3. Given.
4. $m\angle 1 = m\angle 2$.	4. Division postulate: Halves of equal quantities are equal.

We will deal with halves of congruent segments and halves of congruent angles in the same way that we deal with halves of segments whose lengths are equal and halves of angles whose measures are equal. See how this is illustrated in the following example:

EXAMPLE 2 (see Fig. 2–32).

Given: $\overline{AB} \cong \overline{DC}$.
$AF = \frac{1}{2}AB$.
$EC = \frac{1}{2}DC$.

To prove: $\overline{AF} \cong \overline{EC}$.

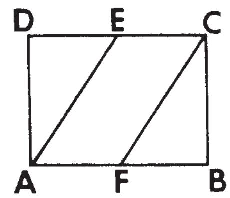


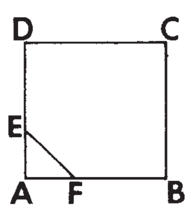
Fig. 2–32

Proof:

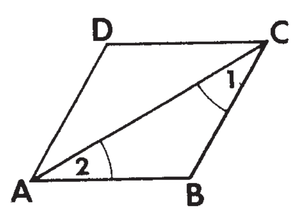
Statements	*Reasons*
1. $\overline{AB} \cong \overline{DC}$.	1. Given.
2. $AF = \frac{1}{2}AB$.	2. Given.
3. $EC = \frac{1}{2}DC$.	3. Given.
4. $\overline{AF} \cong \overline{EC}$.	4. Division postulate: Halves of congruent segments are congruent segments.

EXERCISES

In 1–5, (*a*) state the postulate or postulates that can be used to show that the conclusion is valid and (*b*) write a formal proof.



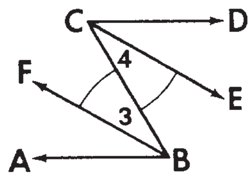
Ex. 1



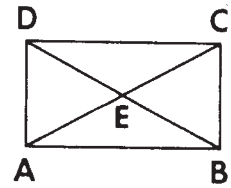
Ex. 2

1. *Given:* $AD = AB$.
$AE = \frac{AD}{3}$.
$AF = \frac{AB}{3}$.
Prove: $AE = AF$.

2. *Given:* $m\angle DAB = m\angle DCB$.
$m\angle 2 = \frac{1}{2}m\angle DAB$.
$m\angle 1 = \frac{1}{2}m\angle DCB$.
Prove: $m\angle 1 = m\angle 2$.



Ex. 3

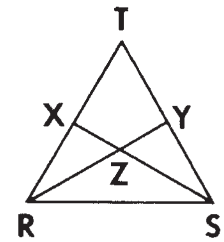


Ex. 4

3. *Given:* $m\angle DCB = m\angle ABC$.
$m\angle 4 = \frac{1}{2} m\angle DCB$.
$m\angle 3 = \frac{1}{2} m\angle ABC$.
Prove: $m\angle 3 = m\angle 4$.

4. *Given:* $\overline{BD} \cong \overline{AC}$.
$BE = \frac{1}{2} BD$.
$AE = \frac{1}{2} AC$.
Prove: $\overline{BE} \cong \overline{AE}$.

5. *Given:* $\angle SRX \cong \angle RSY$.
$m\angle SRY = \frac{1}{2} m\angle SRX$.
$m\angle RSX = \frac{1}{2} m\angle RSY$.
Prove: $\angle SRY \cong \angle RSX$.



Ex. 5

Powers Postulate

***Postulate 10.* If $a = b$, then $a^2 = b^2$.**

We can restate the powers postulate of equality as follows:

***Postulate 10.* The squares of equal quantities are equal.**

EXAMPLE. If $AB = 10$, then $(AB)^2 = (10)^2$, or $(AB)^2 = 100$.

Roots Postulate

***Postulate 11.* If $a = b$, then $\sqrt{a} = \sqrt{b}$.**

We can restate the roots postulate of equality as follows:

***Postulate 11.* Positive square roots of equal quantities are equal.**

EXAMPLE. If $(AB)^2 = 25$, then $\sqrt{(AB)^2} = \sqrt{25}$, or $AB = 5$.

NOTE. In previous postulates which involve the phrase "equal quantities," some people use the word "equals" instead of "equal quantities."

In our work in Chapters 1 and 2, we have dealt with some geometric postulates in an informal manner. Now formal statements of these postulates will be included among the postulates that follow:

Motion Postulate

***Postulate 12.* A geometric figure may be moved without changing its size or shape. [A geometric figure may be copied.]**

Postulates Involving Lines, Line Segments, Angles, and Circles

Postulate 13. **A straight line segment can be extended indefinitely in both directions.**

EXAMPLE. Straight line segment $\overline{AB}$ can be extended indefinitely to the right and to the left. (See Fig. 2–33.)

Fig. 2–33

Postulate 14. **Through two given points one and only one straight line can be drawn. [Two points determine a straight line.]**

EXAMPLE. Through given points C and D, one and only one straight line can be drawn. (See Fig. 2–34.)

Fig. 2–34

Postulate 15. **Two straight lines cannot intersect in more than one point.**

EXAMPLE. If straight lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect, they cannot intersect in more than the one point, E. (See Fig. 2–35.)

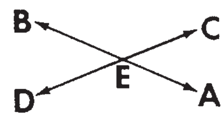


Fig. 2–35

Postulate 16. **One and only one circle can be drawn with any given point as a center and any given line segment as a radius.**

EXAMPLE. Only one circle can be drawn which has point O as its center and a radius equal in length to line segment r. (See Fig. 2–36.)

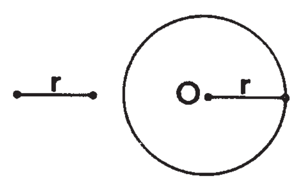


Fig. 2–36

Postulate 17. **At a given point on a given line, one and only one perpendicular can be drawn to the line.**

EXAMPLE. At point P on line $\overleftrightarrow{AB}$, only one line, $\overleftrightarrow{PD}$, can be drawn perpendicular to $\overleftrightarrow{AB}$. (See Fig. 2–37.)

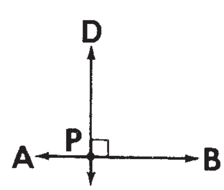


Fig. 2–37

Postulate 18. **From a given point not on a given line, one and only one perpendicular can be drawn to the line.**

EXAMPLE. From point P not on line $\overleftrightarrow{CD}$, only one line, $\overleftrightarrow{PE}$, can be drawn perpendicular to line $\overleftrightarrow{CD}$. (See Fig. 2–38.)

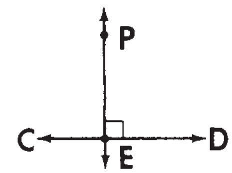


Fig. 2–38

Postulate 19. **For any two distinct points, there is only one positive real number which is called the length of the line segment joining the two points.**

EXAMPLE. For the distinct points A and B, there is one positive real number, represented by AB, which represents the length of $\overline{AB}$ (see Fig. 2–39). The length of $\overline{AB}$ is also called the measure of $\overline{AB}$, or the distance from A to B. We will refer to this postulate as the *distance postulate.*

Fig. 2–39

Postulate 20. **The shortest path between two points is the line segment joining these two points.**

EXAMPLE. Fig. 2–40 pictures three paths that can be taken in going from A to B. The length of line segment $\overline{AB}$ is less than the lengths of the other two paths. The distance between A and B is the measure of the shortest path between A and B.

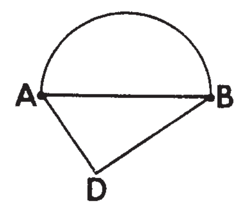


Fig. 2–40

Postulate 21. **A straight line segment has one and only one midpoint.**

EXAMPLE. Straight line segment $\overline{AB}$ has one and only one midpoint, point M. (See Fig. 2–41.)

Fig. 2–41

Postulate 22. **An angle has one and only one bisector.**

EXAMPLE. $\angle ABC$ has one and only one bisector, ray $\overrightarrow{BD}$. (See Fig. 2–42.)

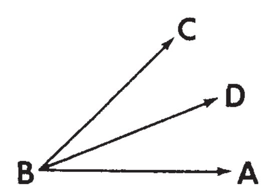


Fig. 2–42

Consider the set of points A, B, C, D, . . . , no two of which are collinear with a given point O. (See Fig. 2–43.) If each of the rays $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$, . . . , is drawn, then the sum of the measures of the consecutive angles thus formed is 360. We shall state this postulate in the following abbreviated form:

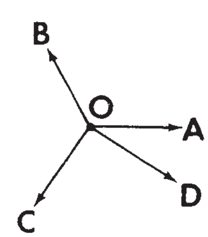


Fig. 2–43

Postulate 23. **The sum of the measures of all the angles about a given point is 360.**

EXAMPLE. The sum of the measures of all the angles about point O, $m\angle AOB + m\angle BOC + m\angle COD + m\angle DOA = 360$. (See Fig. 2–43.)

Consider $\overleftrightarrow{AB}$, which contains a given point P. (See Fig. 2–44.) Let $C, D, \ldots$ be a set of points no two of which are collinear with P and such that they all lie in one of the half-planes formed by $\overleftrightarrow{AB}$. If rays $\overrightarrow{PC}, \overrightarrow{PD}, \ldots$ are drawn, then the sum of the measures of the consecutive angles thus formed is 180. We shall state this postulate in the following abbreviated form:

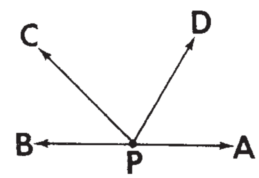


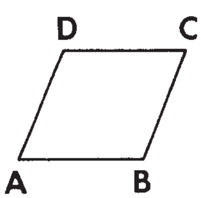
Fig. 2–44

Postulate 24. **The sum of the measures of all the angles about a given point on one side of a given straight line is 180.**

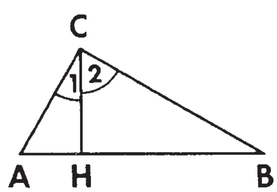
EXAMPLE. The sum of all the angles about point P on one side of line $\overleftrightarrow{AB}$, $m\angle APD + m\angle DPC + m\angle CPB = 180$. (See Fig. 2–44.)

EXERCISES

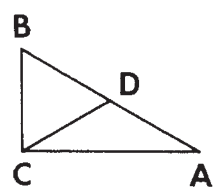
In 1–17, (*a*) state the postulate or postulates that can be used to prove that the conclusion is valid and (*b*) write a formal proof.



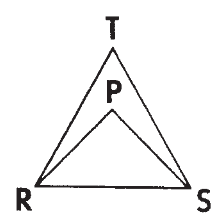
Ex. 1



Ex. 2



Ex. 3

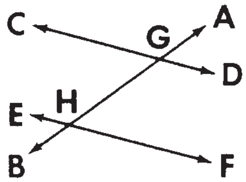


Ex. 4

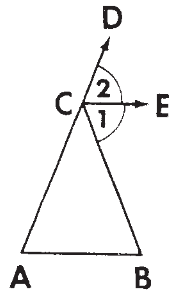
1. If $AB = AD$ and $DC = AD$, then $AB = DC$.
2. If $m\angle 1 + m\angle 2 = 90$ and $m\angle A = m\angle 2$, then $m\angle 1 + m\angle A = 90$.
3. If $\overline{AD} \cong \overline{CD}$ and $\overline{BD} \cong \overline{CD}$, then $\overline{AD} \cong \overline{BD}$.
4. If $\angle TRS \cong \angle TSR$ and $\angle PRT \cong \angle PST$, then $\angle PRS \cong \angle PSR$.

Exercises 5–8 refer to the figures on the next page.

5. If $m\angle AGH = 180$, $m\angle EHF = 180$, and $m\angle AGD = m\angle GHF$, then $m\angle DGH = m\angle GHE$.
6. If $m\angle A = m\angle B$, $m\angle 1 = m\angle B$, and $m\angle 2 = m\angle A$, then $m\angle 1 = m\angle 2$.
7. If $\angle ACG \cong \angle ADE$, and $\angle BCG \cong \angle ADF$, then $\angle 1 \cong \angle 2$.
8. If $EB = FD$, $AB = 2EB$, and $CD = 2FD$, then $AB = CD$.



Ex. 5



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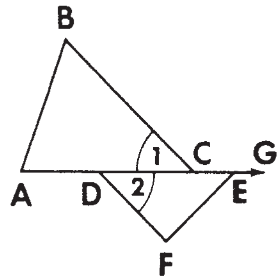

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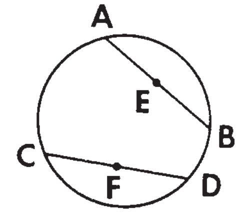

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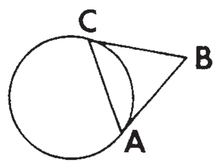

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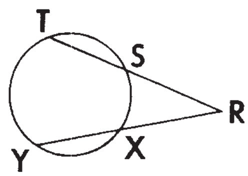
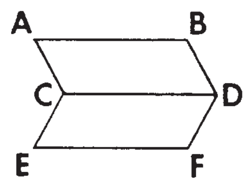


Ex. 10



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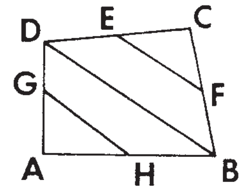
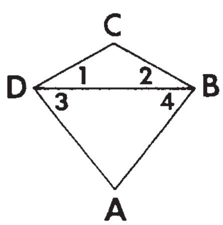
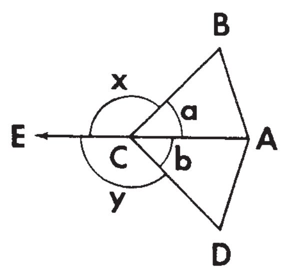


Ex. 12

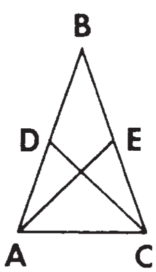
9. If $\overline{AC} \cong \overline{BC}$, and $\overline{BC} \cong \overline{BA}$, then $\overline{AC} \cong \overline{BA}$.
10. If $TS = YX$, and $SR = XR$, then $TR = YR$.
11. If $\overline{AB} \cong \overline{CD}$, and $\overline{EF} \cong \overline{CD}$, then $\overline{AB} \cong \overline{EF}$.
12. If $EF = \frac{1}{2}DB$, and $GH = \frac{1}{2}DB$, then $EF = GH$.



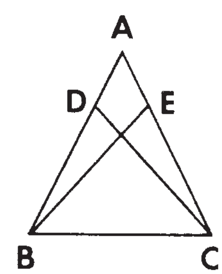
Ex. 13



Ex. 14

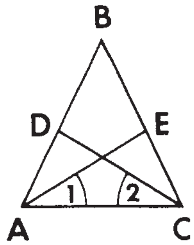


Ex. 15

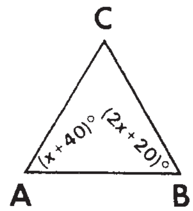


Ex. 16

13. If $\angle 1 \cong \angle 2$, and $\angle 3 \cong \angle 4$, then $\angle CDA \cong \angle CBA$.
14. If $m\angle ECA = 180$, and $m\angle x = m\angle y$, then $m\angle a = m\angle b$.
15. If $BA = BC$, $BD = \frac{1}{2}BA$, and $BE = \frac{1}{2}BC$, then $BD = BE$.
16. If $\overline{AD} \cong \overline{AE}$, and $\overline{DB} \cong \overline{EC}$, then $\overline{AB} \cong \overline{AC}$.



Ex. 17



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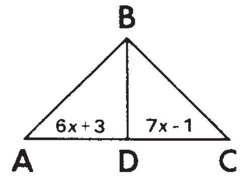
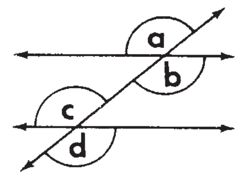


Ex. 19



Ex. 20

17. If $m\angle BAC = m\angle BCA$, $m\angle 1 = \frac{1}{2}m\angle BAC$, and $m\angle 2 = \frac{1}{2}m\angle BCA$, then $m\angle 1 = m\angle 2$.

18. $\angle A \cong \angle C$, and $\angle B \cong \angle C$. (*a*) Prove that $\angle A$ is congruent to $\angle B$. (*b*) Find the number of degrees in $\angle A$ and in $\angle B$.

19. $\overline{AD} \cong \overline{BD}$, and $\overline{DC} \cong \overline{BD}$. (*a*) Prove that $\overline{AD}$ is congruent to $\overline{DC}$. (*b*) Find the length of $\overline{AD}$ and of $\overline{DC}$.

20. *a*. If $m\angle a = m\angle b$, $m\angle c = m\angle d$, and $m\angle b = m\angle c$, prove that $m\angle a = m\angle d$.

b. If $m\angle a = (2y - 60)$, and $m\angle d = (240 - y)$, find the number of degrees in $\angle a$ and in $\angle d$.

8. More Difficult Exercises in Using Postulates and Definitions in Geometric Proofs

In some problems, the hypothesis (*given*), or the conclusion (*to prove*), or both contain words and phrases that we have defined previously. It is important that we know the definitions of these terms so that we will be able to use the given information in developing the proof of the problem.

Example 1 (see Fig. 2–45).

Given: $\overline{RT} \cong \overline{ST}$.
A is the midpoint of $\overline{RT}$.
B is the midpoint of $\overline{ST}$.

To prove: $\overline{RA} \cong \overline{SB}$.

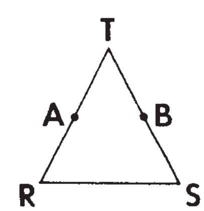


Fig. 2–45

[The proof is given on the next page.]

Proof: *Statements*	*Reasons*
1. $\overline{RT} \cong \overline{ST}$, or $RT = ST$.	1. Given.
2. A is the midpoint of $\overline{RT}$.	2. Given.
3. $\overline{RA} \cong \overline{AT}$, or $RA = \frac{1}{2}RT$.	3. The midpoint of a line segment is the point which divides the segment into two congruent segments.
4. B is the midpoint of $\overline{ST}$.	4. Given.
5. $\overline{SB} \cong \overline{BT}$, or $SB = \frac{1}{2}ST$.	5. The midpoint of a line segment is the point which divides the segment into two congruent segments.
6. $RA = SB$.	6. Division postulate: Halves of equal quantities are equal.
7. $\overline{RA} \cong \overline{SB}$.	7. If two segments are equal in length, they are congruent segments.

We may not assume special relationships that appear to be true in the figure drawn for a particular problem. For example, we may not assume that two line segments in a figure are congruent or are perpendicular to each other merely because they appear to be so in the figure. However, unless otherwise stated, we will assume that lines that appear to be straight lines in a figure actually are straight lines.

EXAMPLE 2 (see Fig. 2–46).

Fig. 2–46

Given: $\overline{AB} \cong \overline{CD}$.

To prove: $\overline{AC} \cong \overline{BD}$.

Proof: *Statements*	*Reasons*
1. $\overline{AB} \cong \overline{CD}$.	1. Given.
2. $\overline{BC} \cong \overline{BC}$.	2. Reflexive property: A segment is congruent to itself.
3. $\overline{AB} + \overline{BC} \cong \overline{CD} + \overline{BC}$, or $\overline{AC} \cong \overline{BD}$.	3. Addition postulate: If congruent segments are added to congruent segments, the sums are congruent segments.

EXAMPLE 3 (see Fig. 2–47).

Given: $\angle ABC \cong \angle DBE$.

To prove: $\angle ABD \cong \angle CBE$.

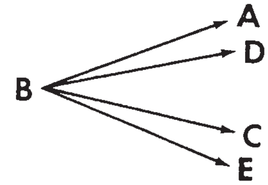
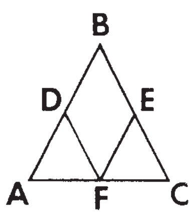


Fig. 2–47

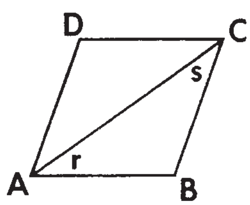
Proof: *Statements*	*Reasons*
1. $\angle ABC \cong \angle DBE$.	1. Given.
2. $\angle DBC \cong \angle DBC$.	2. Reflexive property: An angle is congruent to itself.
3. $\angle ABC - \angle DBC \cong \angle DBE - \angle DBC$, or $\angle ABD \cong \angle CBE$.	3. Subtraction property: If congruent angles are subtracted from congruent angles, the differences are congruent angles.

EXERCISES

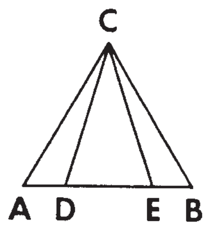
In 1–16, write a formal proof which demonstrates that the conclusion is valid.



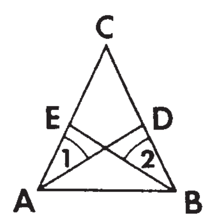
Ex. 1



Ex. 2



Ex. 3

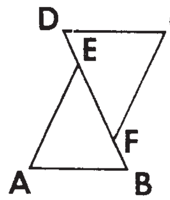


Ex. 4

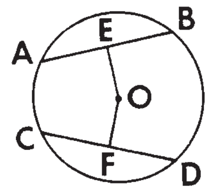
1. If $\overline{AB} \cong \overline{CB}$, $\overline{FD}$ bisects $\overline{AB}$, and $\overline{FE}$ bisects $\overline{CB}$, then $\overline{AD} \cong \overline{CE}$.
2. If $\overline{CA}$ bisects both $\angle DCB$ and $\angle DAB$, and $\angle DCB \cong \angle DAB$, then $\angle r \cong \angle s$.
3. If $\overline{AD} \cong \overline{BE}$, then $\overline{AE} \cong \overline{BD}$.
4. If $\overline{AD}$ bisects $\angle CAB$, $\overline{EB}$ bisects $\angle CBA$, and $\angle CAB \cong \angle CBA$, then $\angle 1 \cong \angle 2$.

Exercises 5–8 refer to the figures on the next page.

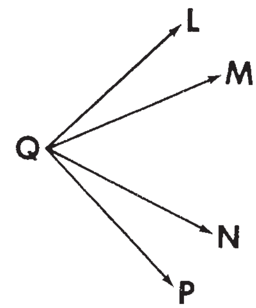
5. If $\overline{DF} \cong \overline{BE}$, then $\overline{DE} \cong \overline{BF}$.
6. If $\overline{AB} \cong \overline{CD}$, $\overline{OE}$ bisects $\overline{AB}$, and $\overline{OF}$ bisects $\overline{CD}$, then $\overline{AE} \cong \overline{CF}$.
7. If $\angle LQM \cong \angle NQP$, then $\angle LQN \cong \angle MQP$.
8. If $\overline{AB} \cong \overline{BC}$, $\overline{CD}$ is a median to $\overline{AB}$, and $\overline{AE}$ is a median to $\overline{BC}$, then $\overline{BD} \cong \overline{BE}$.



Ex. 5



Ex. 6



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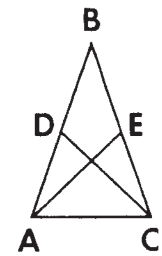
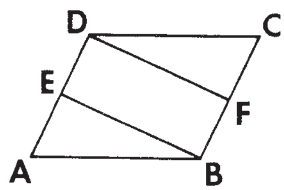


Ex. 8



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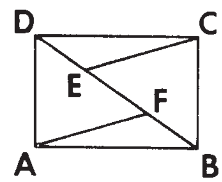
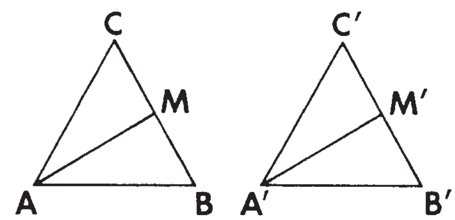
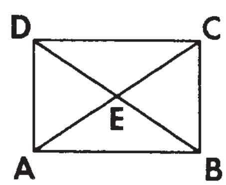


Ex. 10

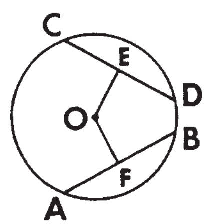


Ex. 11

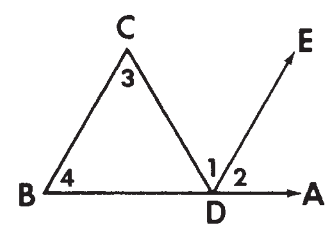
9. If $\overline{AD} \cong \overline{BC}$, E is the midpoint of $\overline{AD}$, and F is the midpoint of $\overline{BC}$, then $\overline{AE} \cong \overline{FC}$.
10. If $\overline{DE} \cong \overline{FB}$, then $\overline{DF} \cong \overline{EB}$.
11. If $\overline{BC} \cong \overline{B'C'}$, $\overline{AM}$ is the median to $\overline{BC}$, and $\overline{A'M'}$ is the median to $\overline{B'C'}$, then $\overline{MB} \cong \overline{M'B'}$.



Ex. 12

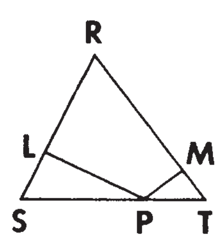


Ex. 13

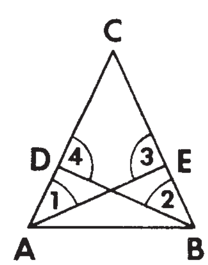


Ex. 14

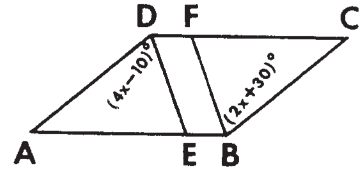
12. If $\overline{AC} \cong \overline{DB}$, and $\overline{AC}$ and $\overline{DB}$ bisect each other, then $\overline{AE} \cong \overline{EB}$.
13. If $\overline{AF} \cong \overline{CE}$, $\overline{OF}$ bisects $\overline{AB}$, and $\overline{OE}$ bisects $\overline{CD}$, then $\overline{AB} \cong \overline{CD}$.
14. If $\overrightarrow{DE}$ bisects $\angle CDA$, $\angle 3 \cong \angle 1$, and $\angle 4 \cong \angle 2$, then $\angle 3 \cong \angle 4$.



Ex. 15



Ex. 16

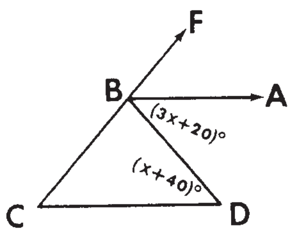


Ex. 17

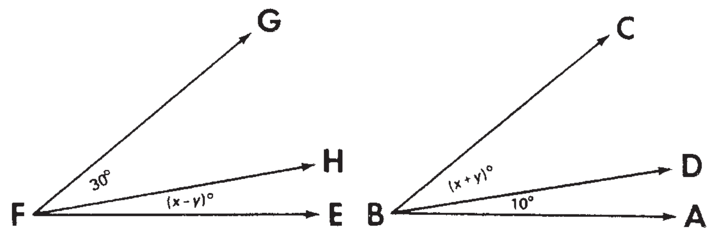
15. If $\angle RLS$ and $\angle RMT$ are straight angles, and if $\overline{PL} \perp \overline{RS}$ and $\overline{PM} \perp \overline{RT}$, then $m\angle PLS = m\angle PMT$.

16. If $m\angle 1 + m\angle 3 + m\angle C = 180$, $m\angle 2 + m\angle 4 + m\angle C = 180$, and $m\angle 3 = m\angle 4$, then $m\angle 1 = m\angle 2$.

17. If $m\angle ADC = m\angle ABC$, $\overline{DE}$ bisects $\angle ADC$, and $\overline{BF}$ bisects $\angle ABC$: (*a*) Prove that $m\angle ADE = m\angle CBF$. (*b*) Find the number of degrees in $\angle ADE$ and in $\angle ABC$.



Ex. 18

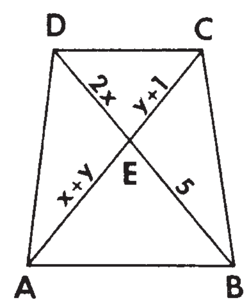


Ex. 19

18. If $\overrightarrow{BA}$ bisects $\angle FBD$, and $m\angle ABD = m\angle BDC$: (*a*) Prove that $m\angle ABF = m\angle BDC$. (*b*) Find the number of degrees in $\angle ABF$.

19. If $\angle EFG \cong \angle ABC$, and $\angle EFH = \angle ABD$: (*a*) Prove that $\angle HFG \cong \angle DBC$. (*b*) Solve for x and y.

20. If $\overline{DB} \cong \overline{AC}$, and $\overline{AE} \cong \overline{EB}$: (*a*) Prove that $\overline{DE} \cong \overline{EC}$. (*b*) Find DE, EC, DB, and AC.



Ex. 20

9. Proving and Using Simple Angle Theorems

We already know that a theorem is a statement proved by deduction. Now we will see how we can use our undefined terms, defined terms, and postulates in proving some simple angle theorems. In theorems 1–8 that follow, the proofs of theorems 1, 3, and 8 are presented; the proofs of theorems 2, 4, 5, 6, and 7 are left to the student.

***Theorem 1.* If two angles are right angles, then they are congruent.**

Given: $\angle ABC$ and $\angle DEF$ are right angles. (See Fig. 2–48.)

To prove: $\angle ABC \cong \angle DEF$.

[The proof is given on the next page.]

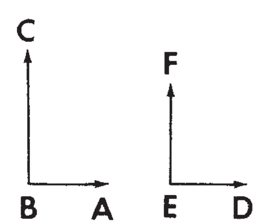


Fig. 2–48

Proof: Statements	Reasons
1. $\angle ABC$ and $\angle DEF$ are right angles.	1. Given.
2. $m\angle ABC = 90$, $m\angle DEF = 90$.	2. A right angle is an angle whose measure is 90.
3. $m\angle ABC = m\angle DEF$.	3. Transitive postulate of equality: If quantities are equal to the same quantity or equal quantities, they are equal to each other.
4. $\angle ABC \cong \angle DEF$.	4. If the measures of two angles are equal, the angles are congruent.

Note that $\angle ABC \cong \angle DEF$ is equivalent to $m\angle ABC = m\angle DEF$.

***Theorem 2.* If two angles are straight angles, then they are congruent.**

In Fig. 2–49, if $\angle ABC$ and $\angle DEF$ are straight angles, then $\angle ABC \cong \angle DEF$.

Fig. 2–49

***Theorem 3.* If two angles are complements of the same angle, then they are congruent.**

Given: $\angle ABD$ is complementary to $\angle CBD$. $\angle EBC$ is complementary to $\angle CBD$. (See Fig. 2–50.)

To prove: $\angle ABD \cong \angle EBC$.

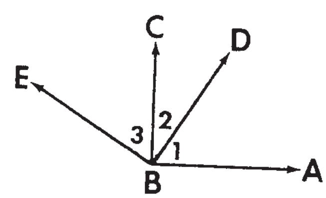


Fig. 2–50

Proof: Statements	Reasons
1. $\angle ABD$ is complementary to $\angle CBD$.	1. Given.
2. $m\angle ABD + m\angle CBD = 90$.	2. If two angles are complementary, the sum of their measures is 90.
3. $\angle EBC$ is complementary to $\angle CBD$.	3. Given.
4. $m\angle EBC + m\angle CBD = 90$.	4. If two angles are complementary, the sum of their measures is 90.

5. $m\angle ABD + m\angle CBD = m\angle EBC + m\angle CBD$.	5. Transitive postulate of equality: If quantities are equal to the same quantity or equal quantities, they are equal to each other.
6. $m\angle CBD = m\angle CBD$.	6. Reflexive property of equality: A quantity is equal to itself.
7. $m\angle ABD = m\angle EBC$.	7. Subtraction postulate of equality: If equal quantities are subtracted from equal quantities, the differences are equal.
8. $\angle ABD \cong \angle EBC$.	8. If the measures of two angles are equal, the angles are congruent.

The following algebraic explanation can also be used to establish the truth of the theorem "If two angles are complements of the same angle, then they are congruent":

If $\angle ABD$ is complementary to $\angle CBD$ (see Fig. 2-51), then $m\angle ABD + m\angle CBD = 90$. Therefore, if the number of degrees contained in $\angle CBD$ is represented by x, then the number of degrees contained in $\angle ABD$ can be represented by $90 - x$. Similarly, if $\angle EBC$ is complementary to $\angle CBD$, the number of degrees contained in $\angle EBC$ can be represented by $90 - x$. Since $90 - x$ represents the number of degrees contained in both $\angle ABD$ and $\angle EBC$, $m\angle ABD = m\angle EBC$. Hence, $\angle ABD \cong \angle EBC$.

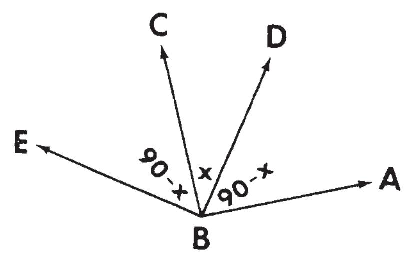


Fig. 2-51

Theorem 4. **If two angles are congruent, their complements are congruent.**

In Fig. 2-52, if $\angle ABD \cong \angle EFH$, $\angle CBD$ is complementary to $\angle ABD$, and $\angle GFH$ is complementary to $\angle EFH$, then $\angle CBD \cong \angle GFH$.

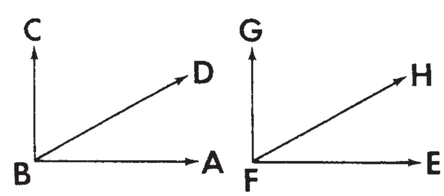


Fig. 2-52

Theorem 5. **If two angles are supplements of the same angle, then they are congruent.**

In Fig. 2–53, if $\angle ABD$ is supplementary to $\angle DBC$, and $\angle EBC$ is supplementary to $\angle DBC$, then $\angle ABD \cong \angle EBC$.

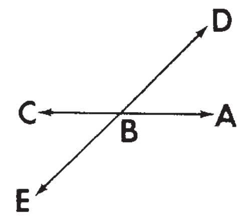


Fig. 2–53

Theorem 6. **If two angles are congruent, then their supplements are congruent.**

In Fig. 2–54, if $\angle ABD \cong \angle EFH$, $\angle CBD$ is supplementary to $\angle ABD$, and $\angle GFH$ is supplementary to $\angle EFH$, then $\angle CBD \cong \angle GFH$.

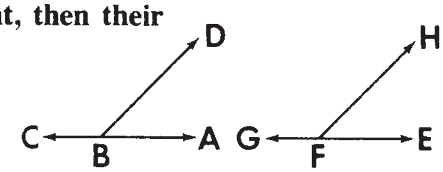


Fig. 2–54

Theorem 7. **If two adjacent angles have their non-common sides on the same straight line, they are supplementary.**

In Fig. 2–55, if $\overrightarrow{BA}$ and $\overrightarrow{BD}$, the non-common sides of adjacent angles ABC and CBD, lie on straight line $\overleftrightarrow{ABD}$, then $\angle ABC$ and $\angle CBD$ are supplementary.

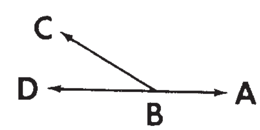


Fig. 2–55

Theorem 8. **If two angles are vertical angles, then they are congruent.**

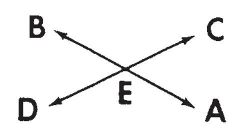


Fig. 2–56

Given: $\angle BEC$ and $\angle AED$ are vertical angles. (See Fig. 2–56.)

To prove: $\angle BEC \cong \angle AED$.

Proof: *Statements*	*Reasons*
1. $\angle BEC$ and $\angle AED$ are vertical angles.	1. Given.
2. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are straight lines that intersect at E.	2. Definition of vertical angles.
3. $m\angle BEC + m\angle AEC = 180$, $m\angle AED + m\angle AEC = 180$.	3. The sum of the measures of all the angles about a point on one side of a straight line is 180.
4. $m\angle BEC + m\angle AEC = m\angle AED + m\angle AEC$.	4. Transitive postulate of equality: If quantities are equal to the same quantity, they are equal to each other.
5. $m\angle AEC = m\angle AEC$.	5. Reflexive property of equality: Any quantity is equal to itself.

6. $m\angle BEC = m\angle AED$.	6. Subtraction postulate: If equal quantities are subtracted from equal quantities, the differences are equal.
7. $\angle BEC \cong \angle AED$.	7. If the measures of two angles are equal, the angles are congruent.

How to Present a Formal Proof

Theorems 1–8 can be used with the undefined terms, the defined terms, and the postulates to deduce or prove new conclusions.

In order to make a formal presentation of the deductive process that is used in proving a desired conclusion from a given set of data, which we call the *hypothesis,* we will:

1. Carefully draw a good *figure* which pictures the data of the theorem or problem. Letter the figure.
2. State the *given,* which is the hypothesis of the theorem, in terms of the lettered figure.
3. State the *to prove,* which is the conclusion of the theorem, in terms of the lettered figure.
4. Present the *proof,* which is the series of logical arguments used in the demonstration. Each step in the proof should consist of a *statement* and its *reason.* A reason may be the *given,* a *definition,* a *postulate,* or a *previously proved theorem.*

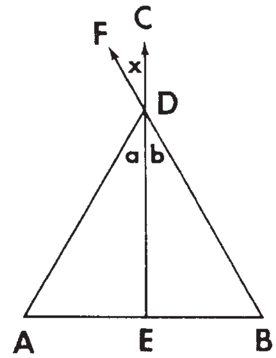
KEEP IN MIND

Two angles may be proved congruent by showing that any one of the following statements is true about them:

1. They are right angles or straight angles.
2. They are complements of the same angle or congruent angles.
3. They are supplements of the same angle or congruent angles.
4. They are vertical angles formed by two intersecting straight lines.

MODEL PROBLEMS

1. Write a formal proof: If $\overleftrightarrow{CE}$ bisects $\angle ADB$, and $\overleftrightarrow{FDB}$ and $\overleftrightarrow{CDE}$ are straight lines, then $\angle a \cong \angle x$.



Given: $\overleftrightarrow{CE}$ bisects $\angle ADB$.
$\overleftrightarrow{FDB}$ and $\overleftrightarrow{CDE}$ are straight lines.

To prove: $\angle a \cong \angle x$.

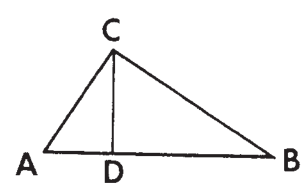
Proof:

Statements	*Reasons*
1. $\overleftrightarrow{CE}$ bisects $\angle ADB$.	1. Given.
2. $\angle a \cong \angle b$.	2. A bisector of an angle divides the angle into two congruent angles.
3. $\overleftrightarrow{FDB}$ and $\overleftrightarrow{CDE}$ are straight lines.	3. Given.
4. $\angle x$ and $\angle b$ are vertical angles.	4. Definition of vertical angles.
5. $\angle b \cong \angle x$.	5. If two angles are vertical angles, then they are congruent.
6. $\angle a \cong \angle x$	6. Transitive property of congruence of angles.

2. Write a formal proof: If $\angle ACB$ is a right angle and $\angle DAC$ is complementary to $\angle ACD$, then $\angle BCD \cong \angle DAC$.

Given: $\angle ACB$ is a right angle.
$\angle DAC$ is complementary to $\angle ACD$.

To prove: $\angle BCD \cong \angle DAC$.

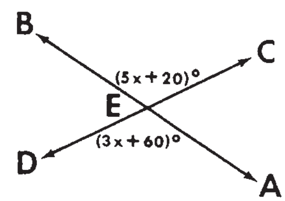


Proof:

Statements	*Reasons*
1. $\angle ACB$ is a right angle.	1. Given.
2. $m\angle ACB = 90$.	2. The measure of a right angle is 90.

3. $m\angle ACB = m\angle ACD + m\angle BCD$.	3. The measure of a whole quantity is equal to the sum of the measures of all its parts.
4. $m\angle ACD + m\angle BCD = 90$.	4. Substitution postulate.
5. $\angle BCD$ is complementary to $\angle ACD$.	5. If the sum of the measures of two angles is 90, the angles are complementary.
6. $\angle DAC$ is complementary to $\angle ACD$.	6. Given.
7. $\angle BCD \cong \angle DAC$.	7. If two angles are complements of the same angle, then they are congruent.

3. If lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E, find: (a) the value of x, (b) $m\angle BEC$, and (c) $m\angle CEA$.



Solution: *a.* Since straight lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E, the two vertical angles that are formed, $\angle BEC$ and $\angle AED$, are congruent. Hence,

1. $m\angle BEC = m\angle AED$
2. $5x + 20 = 3x + 60$
3. $5x - 3x = 60 - 20$
4. $2x = 40$
5. $x = 20$

Answer: (a) The value of x is 20.

(b) $m\angle BEC = 5x + 20 = 5(20) + 20 = 100 + 20 = 120$

Answer: (b) $m\angle BEC = 120$.

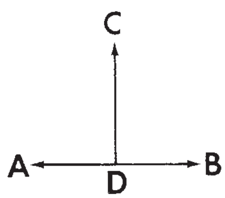
(c) Since $\overleftrightarrow{AB}$ is a straight line, $\angle CEA$ is the supplement of $\angle BEC$.

$$m\angle CEA = 180 - m\angle BEC = 180 - 120 = 60$$

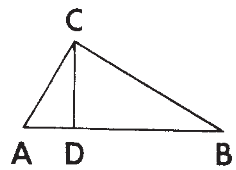
Answer: (c) $m\angle CEA = 60$.

EXERCISES

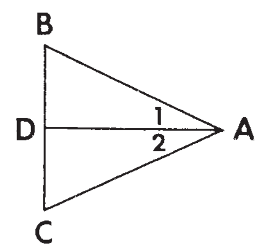
In 1–17, write a formal proof using the given, definitions, postulates, and theorems as the reasons for the statements used in the proof.



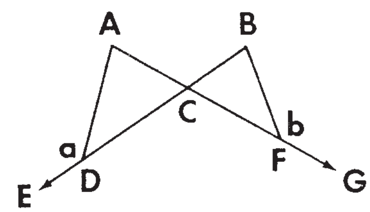
Ex. 1



Ex. 2

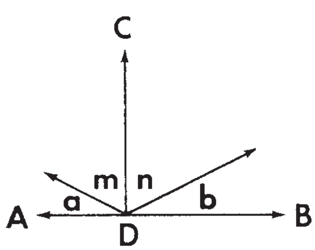


Ex. 3

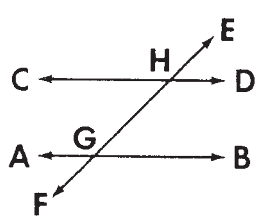


Ex. 4

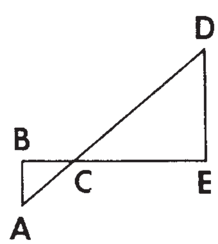
1. If $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$, then $\angle CDA \cong \angle CDB$.
2. If ABC is a triangle with $\angle ACB$ a right angle and $\overline{CD} \perp \overline{AB}$, then $\angle ACB \cong \angle ADC$.
3. If $\angle 1 \cong \angle 2$, $\angle B$ is complementary to $\angle 1$, and $\angle C$ is complementary to $\angle 2$, then $\angle B \cong \angle C$.
4. If $\overleftrightarrow{AG}$ and $\overleftrightarrow{BE}$ are straight lines and $\angle a \cong \angle b$, then $\angle ADC \cong \angle BFC$.



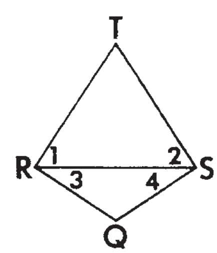
Ex. 5



Ex. 6

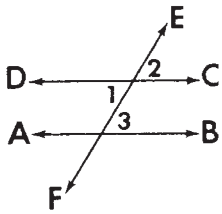


Ex. 7

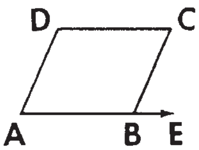


Ex. 8

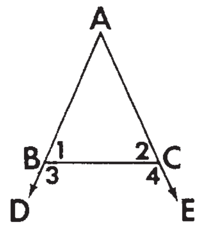
5. If $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$ and $\angle m \cong \angle n$, then $\angle a \cong \angle b$.
6. If $\overleftrightarrow{CD}$ is a straight line and $\angle AGH$ is supplementary to $\angle CHG$, then $\angle GHD \cong \angle AGH$.
7. If $\overline{BE}$ and $\overline{AD}$ intersect at C, $\angle BAC$ is complementary to $\angle ACB$, and $\angle EDC$ is complementary to $\angle ECD$, then $\angle BAC \cong \angle EDC$.
8. If $\overline{TR} \perp \overline{RQ}$, $\overline{TS} \perp \overline{SQ}$, and $\angle 3 \cong \angle 4$, then $\angle 1 \cong \angle 2$.



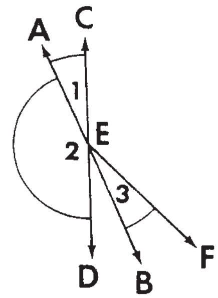
Ex. 9



Ex. 10

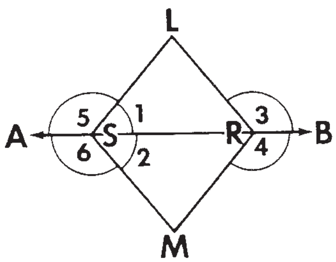


Ex. 11

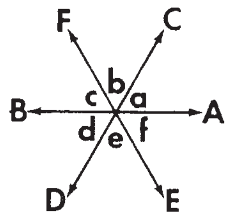


Ex. 12

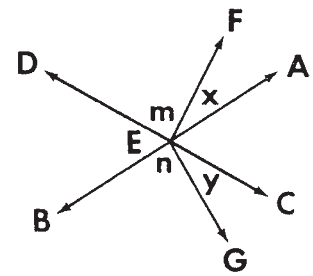
9. If $\overleftrightarrow{AB}$, $\overleftrightarrow{DC}$, and $\overleftrightarrow{EF}$ are straight lines and $\angle 3 \cong \angle 2$, then $\angle 1 \cong \angle 3$.
10. If $\overleftrightarrow{AE}$ is a straight line and $\angle CBE$ is supplementary to $\angle ADC$, then $\angle ADC \cong \angle ABC$.
11. If $\overleftrightarrow{AD}$ and $\overleftrightarrow{AE}$ are straight lines and $\angle 1 \cong \angle 2$, then $\angle 3 \cong \angle 4$.
12. If $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{EF}$ are straight lines and $\angle 1 \cong \angle 3$, then $\angle 3$ is supplementary to $\angle 2$.



Ex. 13



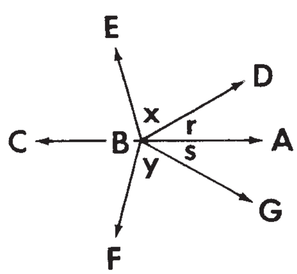
Ex. 14



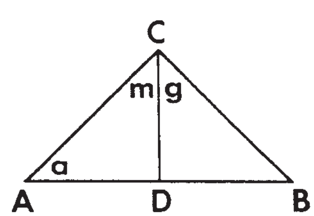
Ex. 15

13. If $\overleftrightarrow{AB}$ is a straight line, $\angle 5 \cong \angle 3$, $\angle 6 \cong \angle 4$, and $\angle 3 \cong \angle 4$, then $\angle 1 \cong \angle 2$.
14. If $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{EF}$ are straight lines and $\angle a \cong \angle b$, $\angle b \cong \angle c$, then $\angle d \cong \angle e$, $\angle e \cong \angle f$.
15. If straight lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E, and $\angle x \cong \angle y$, then $\angle m \cong \angle n$.

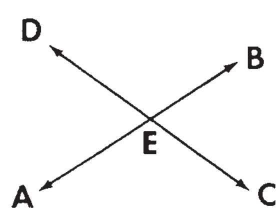
Exercises 16–22 on the next page refer to the following figures:



Ex. 16



Ex. 17



Ex. 18–22

16. If $\overleftrightarrow{ABC}$ is a straight line, $\angle r \cong \angle s$, $\overrightarrow{BE}$ bisects $\angle CBD$, and $\overrightarrow{BF}$ bisects $\angle CBG$, then $\angle x \cong \angle y$.

17. If $\angle ACB$ is a right angle and $\angle a$ is complementary to $\angle g$, then $\angle a \cong \angle m$.

In 18–22, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are straight lines.

18. If $m\angle BEC = 70$, find $m\angle AED$, $m\angle DEB$, and $m\angle AEC$.

19. If $m\angle DEB = 2x + 20$ and $m\angle AEC = 3x - 30$, find $m\angle DEB$, $m\angle AEC$, $m\angle AED$, and $m\angle CEB$.

20. If $m\angle BEC = 5x - 25$ and $m\angle DEA = 7x - 65$, find $m\angle BEC$, $m\angle DEA$, $m\angle DEB$, and $m\angle AEC$.

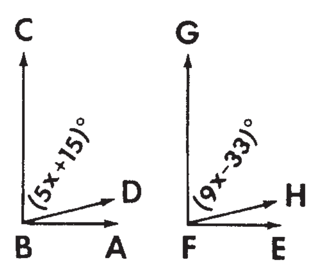
21. If $m\angle BEC = y$, $m\angle DEB = 3x$, and $m\angle DEA = 2x - y$, find $m\angle CEB$, $m\angle BED$, $m\angle DEA$, and $m\angle AEC$.

22. If $m\angle AED = 3r$, $m\angle DEB = 5s + 12$, and $m\angle BEC = r + s + 8$, find $m\angle AED$, $m\angle DEB$, $m\angle BEC$, and $m\angle CEA$.

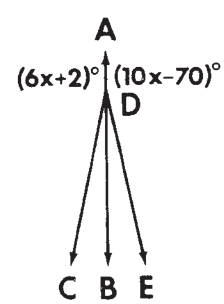
23. $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$ at E. $m\angle AEC = 2x + 30$ and $m\angle DEB = 4x - 50$. Find $m\angle AEC$ and $m\angle DEB$.

24. $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$ at E. $m\angle AED = \frac{3}{2}x + 10$ and $m\angle BEC = x + 36$. Find $m\angle BEC$ and $m\angle CEA$.

25. $\overleftrightarrow{RS}$ intersects $\overleftrightarrow{LM}$ at P. $m\angle RPL = x + y$, $m\angle LPS = 3x + 2y$, $m\angle MPS = 3x - 2y$. (*a*) Solve for x and y. (*b*) Find $m\angle RPL$, $m\angle LPS$, and $m\angle MPS$.



Ex. 26



Ex. 27

26. If $\overleftrightarrow{CB} \perp \overleftrightarrow{BA}$, $\overleftrightarrow{GF} \perp \overleftrightarrow{FE}$, and $\angle ABD \cong \angle EFH$: (*a*) Prove that $\angle CBD \cong \angle GFH$. (*b*) Find $m\angle CBD$ and $m\angle HFE$.

27. If $\overleftrightarrow{AB}$ bisects $\angle CDE$: (*a*) Prove that $\angle ADC \cong \angle ADE$. (*b*) Find the number of degrees in $\angle ADC$ and $\angle BDE$.

28. *Prove:* (*a*) Theorem 2 on page 92 (*b*) Theorem 4 on page 93 (*c*) Theorem 5 on page 94 (*d*) Theorem 6 on page 94 (*e*) Theorem 7 on page 94.

10. Completion Exercises

In 1–15, write a word or expression that, when inserted in the blank, will make the resulting statement true.

1. A quantity may be substituted for its ______ in any expression.
2. If two quantities are equal to the same quantity or equal quantities, they are ______.
3. If equal quantities are subtracted from equal quantities, the ______ are equal.
4. Halves of equal quantities are ______.
5. The whole quantity is equal to the ______ of all its parts.
6. If two angles are complementary to the same angle, they are ______.
7. If $a = b$, and $b = c$, then ______.
8. If two adjacent angles have their non-common sides on a straight line, the angles are ______.
9. If equal quantities are added to equal quantities, the ______ are equal.
10. If the non-common sides of two adjacent angles are perpendicular, the angles are ______.
11. If two angles are vertical angles, they are ______.
12. A statement whose truth is assumed is called a(an) ______.
13. A statement proved by deduction is called a(an) ______.
14. Arriving at a general truth as a result of examining a set of particular examples is called ______ reasoning.
15. When we go from the general to the particular, we are engaging in ______ reasoning.

11. True-False Exercises

In 1–10, if the statement is always true, write *true;* if the statement is not always true, write *false.*

1. If two angles are supplementary to the same angle, they are supplementary to each other.
2. If two quantities are equal to the same quantity, they are equal to each other.
3. If two angles are complementary, they are congruent.
4. In a postulational system, all terms must be defined.

5. If two adjacent angles are supplementary, their non-common sides lie on a straight line.
6. If line $\overleftrightarrow{AB}$ is the perpendicular bisector of line segment $\overline{CD}$, then line $\overleftrightarrow{AB}$ must pass through the midpoint of line segment $\overline{CD}$.
7. If $\frac{1}{2}x = 10$, then $x = 5$, because halves of equal quantities are equal.
8. If $x + 9 = 19$, then $x = 10$, because when equal quantities are subtracted from equal quantities, the differences are equal.
9. In a postulational system, there are some statements whose truth is accepted without proof.
10. If the conditional statement, "If p, then q," is true, and if the conclusion, q, is true, then the hypothesis, p, is true.

12. "Always, Sometimes, Never" Exercises

In 1–10, if the blank space in the exercise is replaced by the word *always*, *sometimes*, or *never*, the resulting statement will be true. Select the word which will correctly complete the statement.

1. If two straight lines intersect, the vertical angles formed are ______ congruent.
2. If two straight lines intersect, the vertical angles formed are ______ supplementary.
3. If two angles are complementary to congruent angles, they are ______ congruent.
4. If two angles are complementary to the same angle, they are ______ complementary to each other.
5. If equal quantities are multiplied by equal quantities, the products are ______ equal quantities.
6. If two adjacent angles have their non-common sides on a straight line, the angles are ______ complementary.
7. A theorem is ______ a statement whose truth is assumed.
8. If two angles are supplementary to the same angle, they are ______ congruent.
9. If two angles are supplementary, they are ______ congruent.
10. If the conditional statement, "If p, then q," is true, and if the hypothesis, p, is true, then the conclusion, q, is ______ true.

13. Multiple-Choice Exercises

In 1–10, write the letter preceding the word or expression that best completes the statement.

1. A postulate is a statement which (*a*) is sometimes to be proved (*b*) is accepted without proof (*c*) is always to be proved.
2. If two straight lines intersect, the vertical angles formed are always (*a*) complementary (*b*) congruent (*c*) supplementary.
3. If two angles are congruent and supplementary, the angles are (*a*) acute angles (*b*) obtuse angles (*c*) right angles.
4. Given the statement: "The whole quantity is equal to the sum of all its parts." This statement is classified in this book as (*a*) a postulate (*b*) a theorem (*c*) a definition.
5. Given the statement: "An acute angle is an angle whose measure is greater than 0 and less than 90." This statement is classified in this book as (*a*) a postulate (*b*) a theorem (*c*) a definition.
6. The complement of an angle whose measure is 50 is an angle whose measure is (*a*) 40 (*b*) 130 (*c*) 50.
7. If the measure of an angle is represented by x, the measure of the supplement of the angle is represented by (*a*) $180 - x$ (*b*) $90 - x$ (*c*) $x - 180$.
8. If the non-common sides of two adjacent angles lie on a straight line, the angles are always (*a*) supplementary (*b*) complementary (*c*) congruent.
9. Given the following statements:

   (1) Tabby is a cat.
   (2) All cats have fur.
   (3) Tabby has fur.

   The correct order in which the statements must be arranged so that the first two will make it possible to deduce the third is (*a*) 3, 1, 2 (*b*) 1, 2, 3 (*c*) 2, 1, 3.
10. In the statement "The bisector of the vertex angle of an isosceles triangle is perpendicular to the base," the hypothesis is (*a*) an isosceles triangle (*b*) perpendicular to the base (*c*) the bisector of the vertex angle of an isosceles triangle.