CHAPTER VI

Proportions Involving Line Segments and Similar Triangles

Up to this point, we have studied the conditions under which polygons, particularly triangles, would be congruent; that is, they would have the same "size" and the same "shape." Now we will study the conditions under which two polygons, particularly triangles, will have the same "shape" but will not necessarily have the same "size." In order to engage in this study, it would be best for us to have a clear understanding of *ratio* and *proportion*, particularly in relation to line segments.

1. Ratio and Proportion

The Meaning of Ratio

If line segment $\overline{AB}$ is 8 feet long and line segment $\overline{CD}$ is 4 feet long, we can compare their lengths by using division. If we divide 8 by 4, we have $\frac{8}{4}$, or $\frac{2}{1}$, which tells us that $\overline{AB}$ is twice as long as $\overline{CD}$. Such a comparison is called a *ratio*.

Definition. **The *ratio of two numbers* a and b, where b is not zero, is the number $\frac{a}{b}$.**

Thus, the ratio of 8 to 4 is $\frac{8}{4}$, which is read "8 to 4" or "8 is to 4." The numbers 8 and 4 are called the *terms* of the ratio.

Keep in mind that we do not find the ratio of two objects. We find the ratio of two numbers which are the measures of the two objects in terms of the *same unit of measure*.

For example, if $\overline{AB}$ has a measure of 1 foot and $\overline{CD}$ has a measure of 1 inch, to find the ratio of AB to CD, we first convert 1 foot to 12 inches and then divide 12 by 1, obtaining $\frac{AB}{CD} = \frac{12}{1}$. The inch is the common unit.

Likewise, if $\angle A$ measures 45 and $\angle B$ is a right angle, to find the ratio of $m\angle A$ to $m\angle B$, we first find that the measure of the right angle is 90 and then divide 45 by 90, obtaining $\frac{m\angle A}{m\angle B} = \frac{45}{90}$ or $\frac{1}{2}$. The degree is the common unit of measure.

However, if in the future we should talk about the *ratio of two angles* or *the ratio of two segments*, we will mean the number which is the quotient of their measures in terms of the same unit of measure.

The ratio of a to b may also be written in the form $a:b$. An advantage of this form is that it can be used to express the comparison among three or more numbers. The statement that three numbers are in the ratio 4:5:9 means that the ratio of the first to the second is 4:5, the ratio of the second to the third is 5:9, and the ratio of the first to the third is 4:9. This does not mean that the numbers must be 4, 5, and 9. There are many sets of numbers whose ratio is 4:5:9. For example, 8, 10, 18; also 12, 15, 27. In general, a set of numbers whose ratio is 4:5:9 can be represented by $4x$, $5x$, and $9x$ where $x \neq 0$.

The Meaning of Proportion

Since the ratio $\frac{4}{12}$ is equal to the ratio $\frac{1}{3}$, we may write $\frac{4}{12} = \frac{1}{3}$. The equation $\frac{4}{12} = \frac{1}{3}$ is called a *proportion*. Another way of writing the proportion $\frac{4}{12} = \frac{1}{3}$ is 4:12 = 1:3, which is read "4 is to 12 as 1 is to 3."

Definition. **A *proportion* is an equation which states that two ratios are equal.**

The proportion $\frac{a}{b} = \frac{c}{d}$, or $a:b = c:d$, is read "a is to b as c is to d," or "a divided by b is equal to c divided by d."

The four numbers a, b, c, and d are called the *terms* of the proportion. The number a is called the first term, b is called the second term, c is called the third term, and d is called the fourth term.

The first and fourth terms, a and d, are called the *extremes* of the proportion. (See Fig. 6–1.)

The second and third terms, b and c, are called the *means* of the proportion.

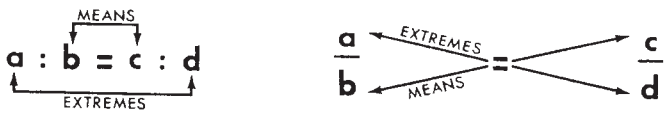


Fig. 6–1

The fact that the four ratios $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, and $\frac{4}{8}$ represent the same number may be expressed in the form of the extended proportion $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}$. In general: $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$ states that the four ratios represent the same number.

The Fourth Proportional

In a proportion in which no terms are equal, the fourth term is called the *fourth proportional* to the other three. In the proportion $a:b = c:d$, the term d is the fourth proportional to a, b, and c.

Geometric Mean and the Third Proportional

If the two means of a proportion are equal, either mean is called the *geometric mean* between the first and fourth terms of the proportion. In the proportion $a:b = b:c$, the term b is the geometric mean between a and c. Also, the fourth term is called the *third proportional* to the other two terms. Thus, c is the third proportional to a and b.

Theorems Involving Proportions

Keep in mind that since a proportion is an equation, all properties of equality can be used to transform a proportion into an equivalent equation. For example, we can transform the proportion $\frac{a}{b} = \frac{c}{d}$ to the equation $ad = bc$ by using the *multiplication property of equality*, multiplying both members by bd.

Theorem 77. **In a proportion, the product of the means is equal to the product of the extremes.**

Thus, if $\frac{4}{8} = \frac{5}{10}$, then $8 \times 5 = 4 \times 10$.

And if $\frac{a}{b} = \frac{c}{d}$, then $bc = ad$.

Theorem 78. **If the product of two numbers (not zero) is equal to the product of two other numbers (not zero), either pair of numbers may be made the means and the other pair may be made the extremes in a proportion.**

Thus if $6 \times 5 = 3 \times 10$, then $\frac{6}{3} = \frac{10}{5}$ or $\frac{3}{6} = \frac{5}{10}$.

And, if $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$ or $\frac{b}{a} = \frac{d}{c}$.

Theorem 79. **If any three terms of one proportion are equal respectively to the three corresponding terms of another proportion, the fourth terms are equal.**

Thus, if $\frac{4}{8} = \frac{x}{16}$ and $\frac{4}{8} = \frac{y}{16}$, then $x = y$.

And, if $\frac{a}{b} = \frac{c}{d}$ and $\frac{a}{b} = \frac{c}{e}$, then $d = e$.

Theorem 80. **If the numerators of the ratios of a proportion are equal (not zero), the denominators are equal; also, if the denominators of the ratios of a proportion are equal, the numerators are equal.**

Thus, if $\frac{a}{b} = \frac{c}{d}$ and $a = c$, then $b = d$.

And, if $\frac{a}{b} = \frac{c}{d}$ and $a = c$, then $a = c$.

Theorem 81. **If four numbers (not zero) are in proportion, they are also in proportion by inversion; that is, the second term is to the first as the fourth is to the third.**

Thus, if $\frac{3}{9} = \frac{4}{12}$, then $\frac{9}{3} = \frac{12}{4}$. And, if $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

Theorem 82. **If four numbers (not zero) are in proportion, they are also in proportion by alternation; that is, the first term is to the third as the second is to the fourth.**

Thus, if $\frac{10}{15} = \frac{8}{12}$, then it is also true that $\frac{10}{8} = \frac{15}{12}$.

And, if $\frac{a}{b} = \frac{c}{d}$, then it is also true that $\frac{a}{c} = \frac{b}{d}$.

Theorem 83. **If four numbers are in proportion, they are also in proportion by addition; that is, the first term plus the second term is to the second term as the third term plus the fourth term is to the fourth term.**

Thus, if $\frac{3}{6} = \frac{4}{8}$, then it is also true that $\frac{3+6}{6} = \frac{4+8}{8}$ or $\frac{9}{6} = \frac{12}{8}$.

And, if $\frac{a}{b} = \frac{c}{d}$, then it is also true that $\frac{a+b}{b} = \frac{c+d}{d}$.

Theorem 84. **If four numbers are in proportion, they are also in proportion by subtraction; that is, the first term minus the second term is to the second term as the third term minus the fourth term is to the fourth term.**

Thus, if $\frac{15}{5} = \frac{12}{4}$, then it is also true that $\frac{15-5}{5} = \frac{12-4}{4}$ or $\frac{10}{5} = \frac{8}{4}$.

And, if $\frac{a}{b} = \frac{c}{d}$, then it is also true that $\frac{a-b}{b} = \frac{c-d}{d}$.

Theorem 85. **In a sequence of equal ratios, the sum of the numerators of the ratios is to the sum of the denominators as any numerator is to its denominator.**

Thus, if $\frac{1}{2} = \frac{3}{6} = \frac{5}{10}$, then it is also true that $\frac{1+3+5}{2+6+10} = \frac{1}{2}$ or $\frac{9}{18} = \frac{1}{2}$.

And, if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then it is also true that $\frac{a+c+e}{b+d+f} = \frac{a}{b}$.

MODEL PROBLEMS

1. Is $\frac{12}{20} = \frac{36}{60}$ a proportion?

Solution: Since $\frac{12}{20} = \frac{3}{5}$ and $\frac{36}{60} = \frac{3}{5}$, the ratios $\frac{12}{20}$ and $\frac{36}{60}$ are equal.

Therefore, $\frac{12}{20} = \frac{36}{60}$ is a proportion.

Answer: Yes.

NOTE. We can also show that $\frac{12}{20} = \frac{36}{60}$ is a proportion by showing that the product of the second and third terms, 20×36, is equal to the product of the first and fourth terms, 12×60.

2. Solve for c in the proportion $18:6 = c:9$.

Solution:

1. $18:6 = c:9$ — In a proportion, the product of the means is equal to the product of the extremes.
2. $6c = 18 \times 9$
3. $6c = 162$
4. $c = 27$ — *Check:* $18:6 \stackrel{?}{=} 27:9$, $3:1 = 3:1$

Answer: $c = 27$.

3. Find the fourth proportional to 3, 4, and 9.

Solution:

1. Let x = the fourth proportional to 3, 4, and 9.
2. Then $3:4 = 9:x$
3. $3x = 36$
4. $x = 12$ — *Check:* $3:4 \stackrel{?}{=} 9:12$, $3:4 = 3:4$

Answer: 12.

4. Find the geometric mean between 4 and 16.

Solution:

1. Let x = the geometric mean between 4 and 16.
2. Then $4:x = x:16$
3. $x^2 = 64$
4. $x = \pm 8$ — Find the square root of both numbers in the preceding equation.

[The solution continues on the next page.]

In our work in geometry, we will restrict our discussion to positive values.

Check: $4:8 \stackrel{?}{=} 8:16$
$1:2 = 1:2$

Answer: $x = 8$.

EXERCISES

1. Use a colon to represent in simplest form the ratio of the first number to the second number:
   a. 8, 2 *b.* 50, 30 *c.* 48, 28 *d.* 15, 45 *e.* 8, 9
2. Express in simplest form the ratio of the first measure to the second measure:
   a. 6 in., 18 in. *b.* 2 ft., 4 in. *c.* 1 right angle, 30°
   d. 2 lb., 8 oz. *e.* 3 dollars, 50 cents *f.* 2 feet, 4 yards
3. Does an equal ratio result when both terms of a given ratio are:
   a. multiplied by the same non-zero number?
   b. divided by the same non-zero number?
   c. increased by the same number?
   d. decreased by the same number?
4. Given $XZ = 4$ and $ZY = 6$. State each of the following ratios:
   a. $XZ:ZY$ *b.* $ZY:XZ$ *c.* $XZ:XY$
   d. $XY:ZY$

Ex. 4

5. B is a point on $\overline{AC}$ such that $\frac{AB}{BC} = \frac{1}{2}$. State each of the following ratios:
   a. $BC:AB$ *b.* $AB:AC$ *c.* $BC:AC$
   d. $AC:BC$

Ex. 5

6. In which of the following may the ratios form a proportion?
   a. $\frac{2}{3}, \frac{24}{36}$ *b.* $\frac{4}{5}, \frac{32}{40}$ *c.* $\frac{3}{4}, \frac{9}{16}$ *d.* $\frac{2}{9}, \frac{10}{54}$ *e.* $\frac{9}{12}, \frac{15}{20}$
7. Use each of the following sets of numbers to form two proportions:
   a. 40, 10, 1, 4 *b.* 4, 6, 18, 12 *c.* 2, 9, 6, 3 *d.* 28, 6, 24, 7
8. Discover which one of the following is a true statement:
   a. $5:10 = 10:20$ *b.* $3:4 = 15:30$ *c.* $12:18 = 36:72$
9. Find and check the value of x in each of the following proportions:
   a. $\frac{x}{10} = \frac{3}{20}$ *b.* $\frac{20}{x} = \frac{10}{24}$ *c.* $\frac{4}{12} = \frac{x}{x+8}$ *d.* $\frac{32}{16} = \frac{21-x}{x}$
   e. $5:x = 8:24$ *f.* $x:10 = 65:5$ *g.* $2r:s = x:t$

10. Express x in terms of r, s, and t:
a. $r:s = t:x$ *b.* $s:2r = t:x$ *c.* $3r:2t = x:4s$

11. Find the fourth proportional to:
a. 1, 4, 5 *b.* 2, 3, 18 *c.* 10, 8, 30 *d.* 4, 18, 16 *e.* r, s, t *f.* $2a$, b, c

12. Find the geometric mean between:
a. 4 and 9 *b.* 2 and 32 *c.* 4 and 25 *d.* $\frac{1}{2}$ and $\frac{1}{8}$ *e.* .3 and 1.2 *f.* c and d

13. Find the third proportional to:
a. 1 and 4 *b.* 4 and 6 *c.* 50 and 10 *d.* .4 and .8 *e.* r and s

14. M is a point on $\overline{LN}$ such that $LM:MN = 3:4$.
a. If $LM = 9$, find MN. *b.* If $MN = 20$, find LN.

Ex. 14

15. Divide a line segment 36 inches long into two parts whose measures are in the ratio 1:8.

16. The measure of an angle and the measure of its complement have the ratio 7:2. Find the number of degrees contained in the angle and in the complement of the angle.

17. The measures of two supplementary angles are in the ratio 1:9. Find the number of degrees contained in the measures of the two angles.

18. If $\frac{r}{s} = \frac{t}{x}$ and $\frac{r}{s} = \frac{t}{y}$, why does $x = y$?

19. Use a theorem of proportions to find the ratio $x:y$ if:
a. $5x = 10y$ *b.* $3x = 4y$ *c.* $2x = y$ *d.* $x = 8y$

20. Write a proportion that can be used to prove:
a. $AE \times EB = CE \times DE$ if $\overline{AE}$, $\overline{EB}$, $\overline{CE}$, and $\overline{DE}$ are line segments.
b. $PC \times PB = PA \times PA$ if $\overline{PC}$, $\overline{PB}$, and $\overline{PA}$ are line segments.

21. Use each given proportion to form a new proportion by inversion:
a. $3:4 = 6:8$ *b.* $5:3 = 50:30$ *c.* $\frac{12}{15} = \frac{36}{45}$ *d.* $\frac{5}{12} = \frac{10}{24}$

22. Use each given proportion to form a new proportion by alternation:
a. $1:2 = 8:16$ *b.* $4:5 = 16:20$ *c.* $\frac{3}{4} = \frac{15}{20}$ *d.* $\frac{20}{5} = \frac{80}{20}$

23. Use the addition or subtraction theorem of proportions to solve each of the following equations:

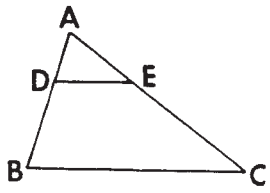
a. $\frac{12-x}{x} = \frac{16}{8}$ *b.* $\frac{20-x}{x} = \frac{6}{4}$ *c.* $\frac{10+x}{x} = \frac{6}{2}$ *d.* $\frac{x+8}{x} = \frac{15}{5}$

24. If $\frac{u}{v} = \frac{w}{x} = \frac{y}{z} = \frac{3}{4}$, find $\frac{u+w+y}{v+x+z}$.

25. *a.* If $\frac{5}{x} = \frac{5}{7}$, find x. *b.* If $\frac{y}{6} = \frac{3}{6}$, find y.

26. Write a proportion in which x is the fourth term and which when solved for x will give:
a. $x = \frac{ab}{c}$ *b.* $x = \frac{rt}{s}$ *c.* $x = \frac{mn}{n}$ *d.* $x = \frac{r^2}{s}$ *e.* $x = \frac{a^2}{b}$

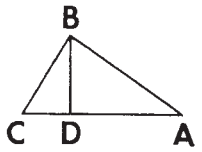
27. In $\triangle ABC$, $\dfrac{AD}{DB} = \dfrac{AE}{EC}$.



Ex. 27

a. Write a new proportion using inversion.
b. Write a new proportion using alternation.
c. Write a new proportion using addition; then name the single line segments whose measures represent the first and third terms of this new proportion.
d. If $AD = 5$, $DB = 15$, and $AE = 8$, find EC.
e. If $AD = 2$, $AE = 6$, and $EC = 18$, find DB.
f. If $DB = 6$, $AE = 12$, and $AC = 36$, find AD.
g. If $AB = 25$, $AD = 10$, and $AC = 30$, find EC.

28. *a. Given:* In $\triangle ABC$, $CD{:}BD = BD{:}DA$. Find AD if $BD = 10$ and $CD = 4$.
b. Given: In $\triangle ABC$, $CD{:}BC = BC{:}CA$. Find BC if $CD = 4$ and $CA = 16$.



Ex. 28

2. Proportions Involving Line Segments

Definition. Two line segments are divided proportionally **when the ratio of the lengths of the segments of one of them is equal to the ratio of the lengths of the segments of the other.**

In Fig. 6–2, $\overleftrightarrow{DE}$ divides $\overline{AB}$ and $\overline{AC}$ proportionally if:

$$\frac{AD}{DB} = \frac{AE}{EC} \quad \left[\frac{\text{length of upper segment}}{\text{length of lower segment}}\right]$$

OR

$$\frac{DB}{AD} = \frac{EC}{AE} \quad \left[\frac{\text{length of lower segment}}{\text{length of upper segment}}\right]$$

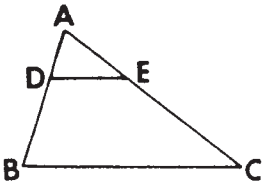


Fig. 6–2

Postulate 37. **If a line is parallel to one side of a triangle and intersects the other two sides, the line divides those sides proportionally.**

If $\overleftrightarrow{DE} \parallel \overline{BC}$ and $\overleftrightarrow{DE}$ intersects $\overline{AB}$ and $\overline{AC}$ (Fig. 6–3), then

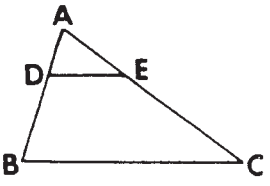


Fig. 6–3

1. $\dfrac{AD}{DB} = \dfrac{AE}{EC}$ $\left[\dfrac{\text{length of upper segment}}{\text{length of lower segment}} = \dfrac{\text{length of upper segment}}{\text{length of lower segment}}\right]$

2. $\dfrac{DB}{AD} = \dfrac{EC}{AE}$ $\left[\dfrac{\text{length of lower segment}}{\text{length of upper segment}} = \dfrac{\text{length of lower segment}}{\text{length of upper segment}}\right]$

If we apply the theorems of proportions that we have learned, we can obtain the following additional proportions:

3. $\dfrac{AD}{AE} = \dfrac{DB}{EC}$ $\left[\dfrac{\text{length of upper segment}}{\text{length of upper segment}} = \dfrac{\text{length of lower segment}}{\text{length of lower segment}}\right]$

4. $\dfrac{AE}{AD} = \dfrac{EC}{DB}$ $\left[\dfrac{\text{length of upper segment}}{\text{length of upper segment}} = \dfrac{\text{length of lower segment}}{\text{length of lower segment}}\right]$

5. $\dfrac{AB}{AD} = \dfrac{AC}{AE}$ $\left[\dfrac{\text{length of whole side}}{\text{length of upper segment}} = \dfrac{\text{length of whole side}}{\text{length of upper segment}}\right]$

6. $\dfrac{AB}{DB} = \dfrac{AC}{EC}$ $\left[\dfrac{\text{length of whole side}}{\text{length of lower segment}} = \dfrac{\text{length of whole side}}{\text{length of lower segment}}\right]$

7. $\dfrac{AD}{AE} = \dfrac{AB}{AC}$ $\left[\dfrac{\text{length of upper segment}}{\text{length of upper segment}} = \dfrac{\text{length of whole side}}{\text{length of whole side}}\right]$

8. $\dfrac{DB}{EC} = \dfrac{AB}{AC}$ $\left[\dfrac{\text{length of lower segment}}{\text{length of lower segment}} = \dfrac{\text{length of whole side}}{\text{length of whole side}}\right]$

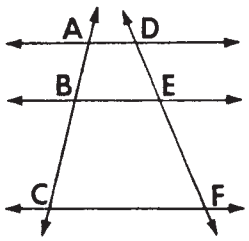


Fig. 6–4

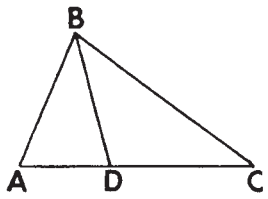


Fig. 6–5

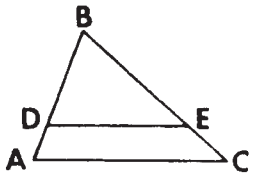


Fig. 6–6

Theorem 86. **Three or more parallel lines intercept proportional segments on any two transversals.**

In Fig. 6–4, if $\overleftrightarrow{AD} \parallel \overleftrightarrow{BE} \parallel \overleftrightarrow{CF}$, then $AB{:}BC = DE{:}EF$.

Theorem 87. **The bisector of an angle of a triangle divides the opposite side into segments whose lengths are proportional to the lengths of the two adjacent sides.**

In Fig. 6–5, if $\overline{BD}$ bisects angle B, then $AD{:}DC = AB{:}CB$.

Postulate 38. **If a line divides two sides of a triangle proportionally, the line is parallel to the third side.**

In Fig. 6–6, if $BD{:}DA = BE{:}EC$, then $\overleftrightarrow{DE} \parallel \overline{AC}$.
Also, if $BA{:}BD = BC{:}BE$, then $\overleftrightarrow{DE} \parallel \overline{AC}$.

MODEL PROBLEMS

1. In triangle RST, a line is drawn parallel to $\overline{ST}$ intersecting $\overline{RS}$ in K and $\overline{RT}$ in L. If $RK = 5$, $KS = 10$, and $RT = 18$, find RL.

Solution: *Method* 1

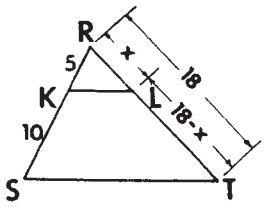
Let $RL = x$ and $LT = 18 - x$.

1. If $\overleftrightarrow{KL} \parallel \overline{ST}$, $\dfrac{RL}{LT} = \dfrac{RK}{KS}$
2. $\dfrac{x}{18 - x} = \dfrac{5}{10}$
3. $10x = 90 - 5x$
4. $15x = 90$
5. $x = 6$

Method 2

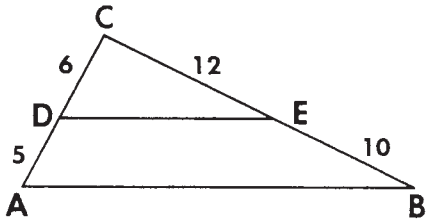
Let $x = RL$.

1. $\dfrac{RL}{RT} = \dfrac{RK}{RS}$
2. $\dfrac{x}{18} = \dfrac{5}{15}$
3. $15x = 90$
4. $x = 6$



Answer: $RL = 6$.

2. In triangle ABC, $CD = 6$, $DA = 5$, $CE = 12$, and $EB = 10$. Is $\overleftrightarrow{DE}$ parallel to $\overline{AB}$?



Solution: Since $\dfrac{CD}{DA} = \dfrac{6}{5}$,

and $\dfrac{CE}{EB} = \dfrac{12}{10} = \dfrac{6}{5}$,

then $\dfrac{CD}{DA} = \dfrac{CE}{EB}$.

Therefore, $\overleftrightarrow{DE}$ divides two sides of $\triangle ABC$, $\overline{CA}$ and $\overline{CB}$, proportionally; and $\overleftrightarrow{DE}$ must be parallel to the third side, $\overline{AB}$.

Answer: Yes.

EXERCISES

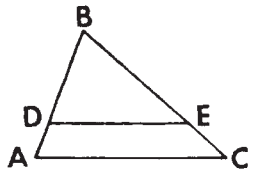
1. If $\overleftrightarrow{DE} \parallel \overline{AC}$, complete the following proportions:

a. $BD:DA = BE:$ _____.

b. $BC:BE =$ _____$:BD$.

c. $AD:EC =$ _____:_____.

d. $BE:BD =$ _____:_____.

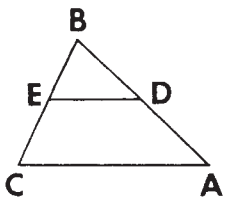


Ex. 1–7

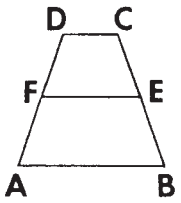
2. If $\overleftrightarrow{DE} \parallel \overline{AC}$ and $BD:DA = 1:2$, find the ratio $BE:EC$.

3. If $BD = 8$, $DA = 4$, $BE = 10$, and $EC = 5$, is $\overleftrightarrow{DE} \parallel \overline{AC}$?

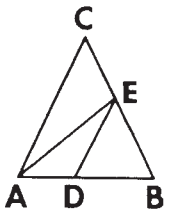
4. If $AD = 9$, $BD = 6$, $EC = 12$, and $BE = 4$, is $\overleftrightarrow{DE} \parallel \overline{AC}$?
5. If $\overleftrightarrow{DE} \parallel \overline{AC}$, $BD = 6$, $DA = 2$, and $BE = 9$, find EC.
6. If $\overleftrightarrow{DE} \parallel \overline{AC}$, $BD{:}DA = 1{:}4$, and $BC = 40$, find EC.
7. If $\overleftrightarrow{DE} \parallel \overline{AC}$, $BE = 7$, $EC = 3$, and $BA = 12$, find BD.
8. In triangle ABC, D is a point on $\overline{AB}$, E is a point on $\overline{AC}$, and $\overline{DE}$ is drawn. If $AB = 8$, $AC = 12$, $DB = 3$, and $EC = 4$, is $\overline{DE} \parallel \overline{BC}$?
9. In triangle ABC, D is a point on $\overline{AB}$, E is a point on $\overline{AC}$, and $\overline{DE}$ is drawn. $AD = 6$, $DB = 4$, $AC = 15$. In order for $\overline{DE}$ to be parallel to $\overline{BC}$, what must be the length of $\overline{EC}$?
10. A line parallel to side $\overline{AB}$ of triangle ABC intersects $\overline{AC}$ at D and $\overline{BC}$ at E. If $DC = 15$, $AD = 5$, and $EC = 18$, find BE.
11. A line parallel to side $\overline{AB}$ of triangle ABC intersects $\overline{CA}$ at D and $\overline{CB}$ at E. If $CD = 4$, $DA = 2$, and $BC = 9$, find CE.
12. Given triangle ABC with a line drawn parallel to $\overline{AC}$ intersecting $\overline{AB}$ at D and $\overline{CB}$ at E. If $AB = 8$, $BC = 12$, and $BD = 6$, find BE.
13. A line parallel to side $\overline{AC}$ of triangle ABC intersects $\overline{AB}$ at D and $\overline{BC}$ at E. If $BD = m$, $DA = n$, and $BE = p$, represent EC in terms of m, n, and p.
14. In triangle ABC, $\overline{BD}$ is the bisector of angle B. If $AB = 6$ and $BC = 8$, find the ratio $AD{:}DC$.
15. The nonparallel sides of a trapezoid measure 12 and 16 respectively. A line parallel to the bases divides the side whose measure is 12 into two segments whose measures are in the ratio 1:3. Find the measures of the segments of the side whose measure is 16.
16. The bisector of an angle of a triangle divides the opposite side into segments whose measures are 4 inches and 3 inches. If the side of the triangle adjacent to the 4-inch segment is 12 inches, find the measure of the side of the triangle adjacent to the 3-inch segment.



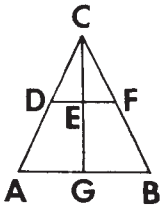
Ex. 17



Ex. 18

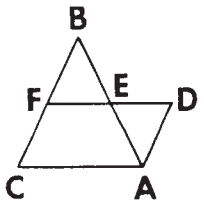


Ex. 19

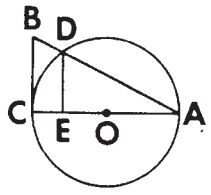


Ex. 20

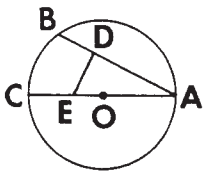
17. If $\angle BDE \cong \angle BAC$, prove that $BE{:}EC = BD{:}DA$.
18. If $\overleftrightarrow{DC} \parallel \overleftrightarrow{FE} \parallel \overleftrightarrow{AB}$, prove that $AF{:}FD = BE{:}EC$.
19. If $\angle DEA \cong \angle CAE$, prove that $AB{:}BD = CB{:}BE$.
20. If $CD{:}DA = CF{:}FB$, prove that $CE{:}CG = CD{:}CA$.



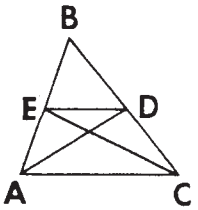
Ex. 21



Ex. 22



Ex. 23



Ex. 24

21. If $ADFC$ is a parallelogram, prove that $BE{:}EA = BF{:}DA$.

22. If in circle O, $\overline{DE} \perp$ diameter $\overline{CA}$, and $\overleftrightarrow{BC}$ is a tangent, prove that $AB{:}BD = AC{:}CE$.

23. If in circle O, $\overline{ED} \perp \overline{AB}$, and $\overline{CA}$ is a diameter, prove that $AD{:}AE = AB{:}AC$.

24. If $\overline{BC} \cong \overline{BA}$, $\overline{CE}$ bisects angle BCA, and $\overline{AD}$ bisects angle BAC, prove that $\overline{ED} \parallel \overline{AC}$.

25. In $\triangle RST$, P is a point on $\overline{RT}$. Through P, a line is drawn parallel to $\overline{ST}$ which intersects $\overline{RS}$ in Y. Through P, a line is drawn parallel to $\overline{RS}$ which intersects $\overline{ST}$ in X. Prove $RY{:}YS = SX{:}XT$.

3. Understanding the Meaning of Similar Polygons

In Fig. 6–7, we see two polygons which have the "same shape" but not the "same size." We call such polygons *similar polygons*. The symbol for the word *similar* is $\sim$.

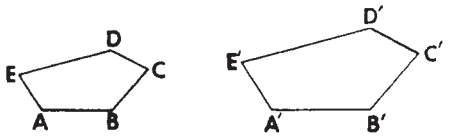


Fig. 6–7

If we study these polygons carefully, we find that they appear to have the same shape because their corresponding angles are congruent and the ratios of the measures of their corresponding sides are equal, (1:2).

We will use this description as a guide in stating a formal definition of similar polygons which will be useful in our work in geometry. Our definition of similar polygons will follow the same pattern that was used in defining congruent polygons.

Definition. Two polygons are similar **if there is a one-to-one correspondence between their vertices such that:**

1. All pairs of corresponding angles are congruent.
2. The ratios of the measures of all pairs of corresponding sides are equal.

In this definition, it is to be understood that the three phrases (1) "a one-to-one correspondence between the vertices of the polygons," (2) "cor-

responding angles of the two polygons," and (3) "corresponding sides of the two polygons" have the same meanings that they had when we dealt with congruent polygons.

In Fig. 6–7, polygon $ABCDE$ is similar to polygon $A'B'C'D'E'$, symbolized polygon $ABCDE \sim$ polygon $A'B'C'D'E'$, if:

1. $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$, $\angle D \cong \angle D'$, $\angle E \cong \angle E'$

AND

2. $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \dfrac{DE}{D'E'} = \dfrac{EA}{E'A'}$

NOTE. When the ratios of the measures of the corresponding sides of two polygons are equal, we say that "the corresponding sides of the two polygons are in proportion."

It is possible for two polygons to have their corresponding sides in proportion and yet not be similar.

Thus, in Fig. 6–8, polygons $ABCD$ and $A'B'C'D'$ have their corresponding sides in proportion. Yet the polygons are not similar because their corresponding angles are not congruent.

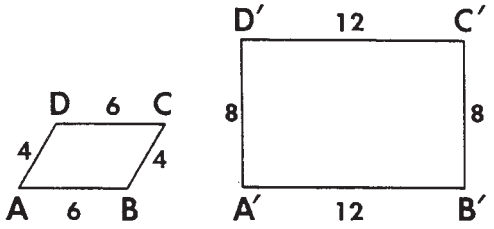


Fig. 6–8

It is also possible for two polygons to have their corresponding angles congruent and yet not be similar.

Thus, in Fig. 6–9, polygons $WXYZ$ and $W'X'Y'Z'$ have their corresponding angles congruent. Yet the polygons are not similar because their corresponding sides are not in proportion.

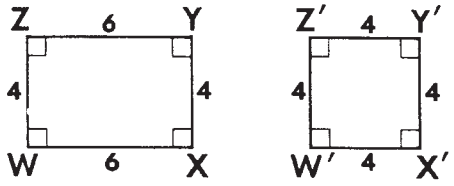


Fig. 6–9

We see that in order for two polygons to be similar, two conditions must be satisfied:

1. All pairs of corresponding angles must be congruent.
2. Their corresponding sides must be in proportion; that is, the ratios of the measures of their corresponding sides must be equal.

Since a definition is reversible, it follows that when two polygons are similar:

1. All pairs of corresponding angles are congruent.
2. Their corresponding sides are in proportion; that is, the ratios of the measures of their corresponding sides are equal.

Note that since triangles are polygons, it follows that if two triangles are similar, their corresponding sides are in proportion, that is, the ratios of the measures of their corresponding sides must be equal.

Definition. **The *ratio of similitude* of two similar polygons is the ratio of the measures of any two corresponding sides.**

If polygon $ABCD$ and polygon $A'B'C'D'$ are similar (Fig. 6–10), the ratio of the measures of any two corresponding sides, or the ratio of similitude, is 2:1.

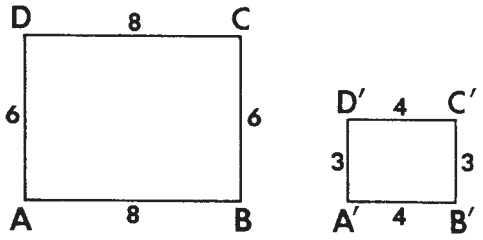


Fig. 6–10

Properties of Similarity

The relationship of similarity of two polygons, like the relationship of congruence of two polygons, gives rise to equations involving the measures of angles and the measures of line segments. Hence, in dealing with the relationship of similarity we may use the following postulates:

THE REFLEXIVE PROPERTY OF SIMILARITY

Postulate 39. **Polygon *ABCD* ~ polygon *ABCD*.**

THE SYMMETRIC PROPERTY OF SIMILARITY

Postulate 40. **If polygon *ABCD* ~ polygon *EFGH*, then polygon *EFGH* ~ polygon *ABCD*.**

THE TRANSITIVE PROPERTY OF SIMILARITY

Postulate 41. **If polygon *ABCD* ~ polygon *EFGH*, and polygon *EFGH* ~ polygon *WXYZ*, then polygon *ABCD* ~ polygon *WXYZ*.**

KEEP IN MIND

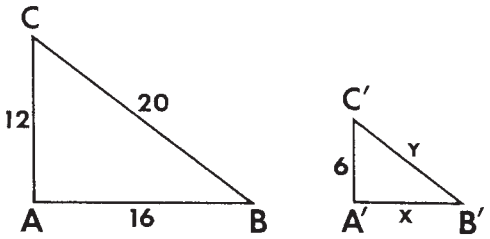
1. To prove that angles are congruent, show that they are corresponding angles of similar polygons.
2. To prove that the measures of line segments are in proportion, show that the line segments are corresponding sides of similar polygons.

MODEL PROBLEM

Triangle $ABC \sim$ triangle $A'B'C'$, and $A'C'$ corresponds to AC.

a. Find the ratio of similitude.

b. Find x and y.



Solution:

a. The ratio of similitude of two similar polygons is the ratio of the measures of any two corresponding sides.

$$\text{Ratio of similitude} = \frac{AC}{A'C'} = \frac{12}{6} = \frac{2}{1}.$$

Answer: 2:1.

b. *Method 1*

Since the ratio of similitude of $\triangle ABC$ to $\triangle A'B'C'$ is 2:1, each side of $\triangle ABC$ is 2 times as long as its corresponding side in $\triangle A'B'C'$, and each side of $\triangle A'B'C'$ is $\frac{1}{2}$ as long as its corresponding side in $\triangle ABC$. Therefore,

$A'B' = \frac{1}{2}AB$, or $x = \frac{1}{2}(16) = 8$

$B'C' = \frac{1}{2}BC$, or $y = \frac{1}{2}(20) = 10$

Answer: $x = 8$, $y = 10$.

Method 2

Since the corresponding sides of similar polygons are in proportion,

1.	$\frac{AC}{A'C'} = \frac{AB}{A'B'}$ and	$\frac{AC}{A'C'} = \frac{BC}{B'C'}$
2.	$\frac{12}{6} = \frac{16}{x}$	$\frac{12}{6} = \frac{20}{y}$
3.	$12x = 96$	$12y = 120$
4.	$x = 8$	$y = 10$

EXERCISES

1. Are two congruent polygons always similar? Why?
2. What is the ratio of similitude of two congruent polygons?
3. Are all similar polygons congruent? Why?
4. What must be the ratio of similitude of two similar polygons in order for them to be congruent polygons?
5. Are all squares similar? Why?
6. Are all rectangles similar? Why?

7. Are all rhombuses similar? Why?
8. Can a quadrilateral be similar to a hexagon? Why?
9. The sides of a triangle measure 4, 8, and 10. If the smallest side of a similar triangle measures 12, find the measures of the remaining sides of this triangle.
10. The sides of a quadrilateral measure 12, 18, 20, and 16. The longest side of a similar quadrilateral measures 5. Find the measures of the remaining sides of this quadrilateral.
11. In two similar triangles, two corresponding sides measure 4 inches and 2 feet. Find the ratio of similitude of the two triangles.
12. Triangle $ABC \sim$ triangle $A'B'C'$, and their ratio of similitude is 1:3. If the measures of the sides of triangle ABC are represented by a, b, and c, represent the measures of the sides of triangle $A'B'C'$.
13. The ratio of similitude of two similar quadrilaterals is 2:1. If the measures of the sides of the larger quadrilateral are represented by w, x, y, and z, represent the measures of the sides of the smaller quadrilateral.
14. Rhombus $ABCD$ has a 60° angle and a side 5 inches in length. Rhombus $A'B'C'D'$ has a 120° angle and a side 10 inches in length. Prove that rhombus $ABCD$ is similar to rhombus $A'B'C'D'$.
15. Prove that any two equiangular triangles are similar.
16. Prove that any two equilateral triangles are similar.

4. Proving Triangles Similar

Since triangles are polygons, we can prove that two triangles are similar by showing that they satisfy the two conditions required in similar polygons. See how this is done in the following discussion:

Consider $\triangle ABC$ and $\triangle A'B'C'$ (Fig. 6–11) in which vertex A corresponds to vertex A', making $\angle A$ and $\angle A'$ a pair of *corresponding angles*, vertex B corresponds to vertex B', making $\angle B$ and $\angle B'$ a pair of *corresponding angles*, and vertex C corresponds to vertex C', making $\angle C$ and $\angle C'$ a pair of *corresponding angles*. Since $m\angle A = 53$ and $m\angle A' = 53$, $\angle A \cong \angle A'$. Since $m\angle B = 37$ and $m\angle B' = 37$, $\angle B \cong \angle B'$. Since $m\angle C = 90$ and $m\angle C' = 90$, $\angle C \cong \angle C'$. Therefore, all pairs of corresponding angles in $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

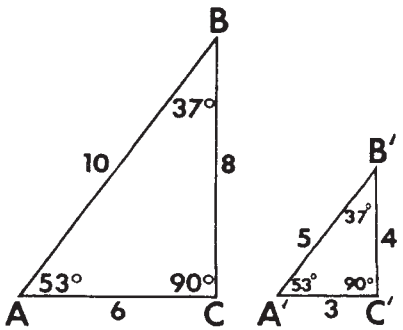


Fig. 6–11

The sides of the two triangles whose endpoints are corresponding vertices in the triangle are *corresponding sides* of the triangles. Therefore, in $\triangle ABC$ and $\triangle A'B'C'$, $\overline{AB}$ and $\overline{A'B'}$ are a pair of corresponding sides, $\overline{AC}$ and $\overline{A'C'}$ are a pair of corresponding sides, and $\overline{BC}$ and $\overline{B'C'}$ are a pair of corresponding sides. In each case, the pair of corresponding sides is opposite a pair of corresponding angles. For example, corresponding sides $\overline{AB}$ and $\overline{A'B'}$ are opposite the corresponding angles C and C' respectively.

Notice that if the measures indicated on the sides of triangles ABC and $A'B'C'$ are correct to the nearest integer, then $\frac{AB}{A'B'}=\frac{10}{5}=\frac{2}{1}$, $\frac{AC}{A'C'}=\frac{6}{3}=\frac{2}{1}$, and $\frac{BC}{B'C'}=\frac{8}{4}=\frac{2}{1}$. Thus, $\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}$. In other words, the corresponding sides of triangle ABC and triangle $A'B'C'$ are in proportion. Therefore, by the definition of similar polygons, $\triangle ABC \sim \triangle A'B'C'$.

However, there are shorter methods of proving triangles similar. These methods are indicated in the theorems and corollaries that follow. In stating these theorems and corollaries, we will follow the same practice that we employed in stating the congruence theorems. In each case, we will understand that a correspondence exists between the vertices of the two triangles for which the congruences and the proportions stated in the hypothesis are true statements.

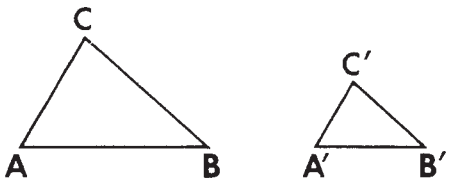


Fig. 6–12

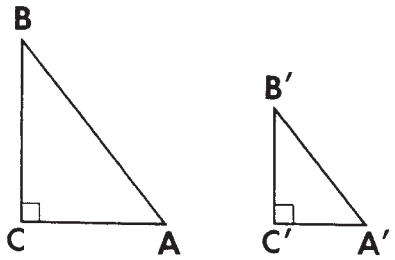


Fig. 6–13

Theorem 88. **Two triangles are similar if three angles of one triangle are congruent to three corresponding angles of the other. [a.a.a. ≅ a.a.a.]**

[The proof for this theorem appears on pages 762–763.]

In $\triangle ABC$ and $\triangle A'B'C'$ (Fig. 6–12), if $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$, then $\triangle ABC \sim \triangle A'B'C'$.

Corollary T88–1. **Two triangles are similar if two angles of one triangle are congruent to two corresponding angles of the other. [a.a. ≅ a.a.]**

In $\triangle ABC$ and $\triangle A'B'C'$ (Fig. 6–12), if $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, then $\triangle ABC \sim \triangle A'B'C'$.

Corollary T88–2. **Two right triangles are similar if an acute angle of one triangle is congruent to an acute angle of the other.**

In right $\triangle ABC$ and $\triangle A'B'C'$ (Fig. 6–13), if acute $\angle A \cong$ acute $\angle A'$, then right $\triangle ABC \sim$ right $\triangle A'B'C'$.

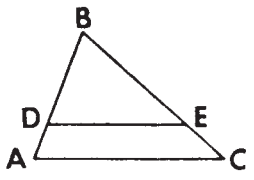


Fig. 6–14

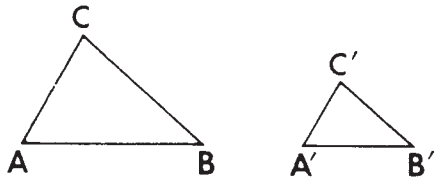


Fig. 6–15

Corollary T88–3. **A line that is parallel to one side of a triangle and intersects the other two sides in different points cuts off a triangle similar to the given triangle.**

In $\triangle ABC$ (Fig. 6–14), if $\overleftrightarrow{DE} \parallel \overline{AC}$, then $\triangle DBE \sim \triangle ABC$.

Theorem 89. **Two triangles are similar if an angle of one triangle is congruent to an angle of the other and the including sides are in proportion.**

In $\triangle ABC$ and $\triangle A'B'C'$ (Fig. 6–15), if $\angle C \cong \angle C'$ and the sides including these angles are in proportion, or $\frac{AC}{A'C'} = \frac{BC}{B'C'}$, then $\triangle ABC \sim \triangle A'B'C'$.

Theorem 90. **Two triangles are similar if their corresponding sides are proportional.**

In $\triangle ABC$ and $\triangle A'B'C'$ (Fig. 6–15), if the corresponding sides are proportional, or $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$, then $\triangle ABC \sim \triangle A'B'C'$.

Methods of Proving Triangles Similar

To prove that two triangles are similar:

1. Use the definitions of similar polygons.

OR

2. Use any one of the preceding three theorems or their corollaries.

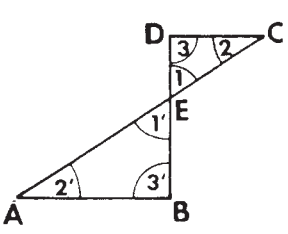
MODEL PROBLEM

Given: Lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect at E. $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$.

a. Prove $\triangle ABE \sim \triangle CDE$.

b. Write any proportion which would be true in the two triangles.

c. If $DE = 10$, $BE = 15$, and $CE = 20$, find AE.



a. Plan: Prove that two angles in $\triangle ABE$ are congruent to two corresponding angles of $\triangle CDE$.

Proof: *Statements*	*Reasons*
1. $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are straight lines.	1. Given.
2. $\angle 1$ and $\angle 1'$ are vertical angles.	2. Definition of vertical angles.
3. $\angle 1 \cong \angle 1'$. (a. $\cong$ a.)	3. If two angles are vertical angles, they are congruent.
4. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.	4. Given.
5. $\angle 2 \cong \angle 2'$. (a. $\cong$ a.)	5. If parallel lines are cut by a transversal, the alternate interior angles are congruent.
6. $\triangle ABE \sim \triangle CDE$.	6. a.a. $\cong$ a.a.

b. Plan: Select pairs of sides which are opposite congruent angles. These sides must be in proportion because corresponding sides of similar triangles are in proportion.

$$\frac{AE \text{ (opposite } \angle 3')}{CE \text{ (opposite } \angle 3)} = \frac{BE \text{ (opposite } \angle 2')}{DE \text{ (opposite } \angle 2)}$$

c.

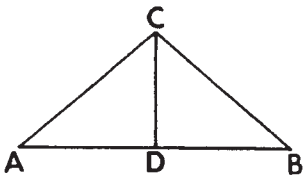
1. Since $AE{:}CE = BE{:}DE$, Let $AE = x$.
2. $x{:}20 = 15{:}10$
3. $10x = 300$
4. $x = 30$

Answer: $AE = 30$.

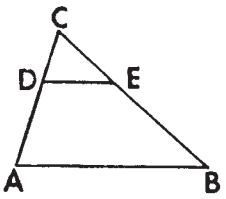
EXERCISES

1. In triangle ABC, $m\angle A = 40$ and $m\angle B = 30$.
   In triangle EDF, $m\angle D = 30$ and $m\angle E = 40$.
   Is triangle ABC similar to triangle EDF? Why?
2. In triangle RST, $m\angle R = 90$ and $m\angle S = 40$.
   In triangle ZXY, $m\angle X = 40$ and $m\angle Y = 50$.
   Is triangle $RST \sim$ triangle ZXY?
3. In triangle ABC, $AC = 4$, $AB = 5$, and $m\angle A = 40$.
   In triangle RTS, $RT = 10$, $RS = 8$, and $m\angle R = 40$.
   Prove that $\triangle ABC \sim \triangle RTS$.

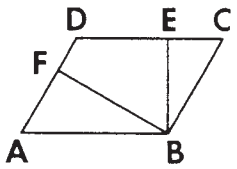
4. In triangle ABC, $AC = 5$, $AB = 4$, and $BC = 6$. In triangle STR, $ST = 8$, $RT = 12$, and $RS = 10$. Prove that $\triangle ABC \sim \triangle STR$.



Ex. 5

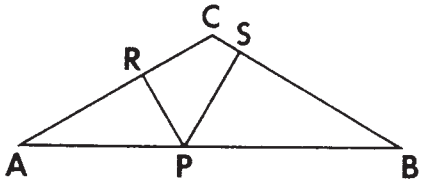


Ex. 6

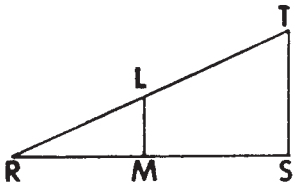


Ex. 7

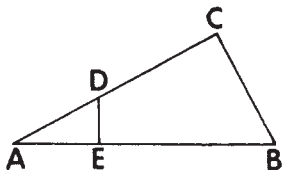
5. In triangle ABC, $\overline{AC} \cong \overline{CB}$ and $\overline{CD}$ bisects angle C. (*a*) Prove $\triangle ACD \sim \triangle BCD$. (*b*) Write a proportion involving the measures of the sides of these triangles.
6. In triangle ABC, $\overline{DE} \parallel \overline{AB}$. (*a*) Prove that $\triangle ABC \sim \triangle DEC$. (*b*) Write a proportion involving the measures of the sides of these triangles. (*c*) If $CD = 6$, $CA = 18$, and $DE = 2$, find AB.
7. In $\square ABCD$, $\overline{BE} \perp \overline{DC}$, and $\overline{BF} \perp \overline{AD}$. (*a*) Prove $\triangle BAF \sim \triangle BCE$. (*b*) Write a proportion involving the measures of the sides of these triangles. (*c*) If $AB = 16$, $BC = 12$, and $FB = 14$, find EB.
8. In $\triangle ABC$, $\overline{AC} \cong \overline{CB}$, $\overline{PR} \perp \overline{AC}$, $\overline{PS} \perp \overline{BC}$. (*a*) Prove $\triangle APR \sim \triangle BPS$. (*b*) Write a proportion involving the measures of the sides of these triangles. (*c*) If $PR = 6$, $PS = 8$, and $AB = 28$, find AP and PB.



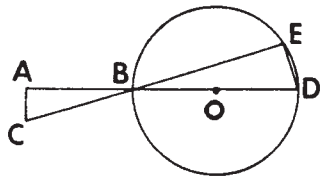
Ex. 8



Ex. 9



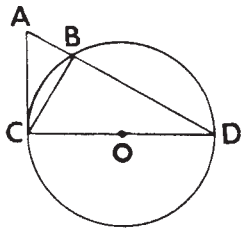
Ex. 10



Ex. 11

9. $\overline{TS} \perp \overline{RS}$ and $\overline{LM} \perp \overline{RS}$. (*a*) Prove $\triangle LMR \sim \triangle TSR$. (*b*) Write a proportion involving the measures of the sides of these triangles. (*c*) If $LM = 6$, $TS = 9$, and $MS = 4$, find RM.
10. In triangle ABC, $\overline{BC} \perp \overline{AC}$, and $\overline{DE} \perp \overline{AB}$. (*a*) Prove $\triangle ABC \sim \triangle ADE$. (*b*) Write a proportion involving the measures of the sides of these triangles. (*c*) If $DE = 5$, $AD = 6$, and $AB = 18$, find BC.
11. In circle O, $\overline{BD}$ is a diameter. $\overleftrightarrow{AD}$ and $\overleftrightarrow{CE}$ are straight lines. $\overleftrightarrow{CA} \perp \overleftrightarrow{AD}$. (*a*) Prove $\triangle ABC \sim \triangle EBD$. (*b*) Write a proportion involving the measures of the sides of these triangles. (*c*) If $AC = 6$, $ED = 12$, $BC = x$, and $BD = x + 10$, find BC and BD.

12. In circle O, $\overline{CD}$ is a diameter and $\overleftrightarrow{AC}$ is a tangent. (*a*) Prove $\triangle BCD \sim \triangle CAD$. (*b*) Write a proportion involving the measures of the sides of these triangles. (*c*) If $AB = 18$ and $BD = 32$, find CD.



Ex. 12

13. Prove that two equilateral triangles are similar.

14. Prove that in a circle if chords $\overline{AB}$ and $\overline{CD}$ intersect in E, triangle AEC is similar to triangle DEB.

15. Prove that the altitude drawn to the hypotenuse of a right triangle divides the triangle into two similar triangles.

16. In triangle ABC, all of whose angles are acute, altitudes $\overline{BD}$ and $\overline{CE}$ intersect at F. Prove that triangles BEF and CEA are similar.

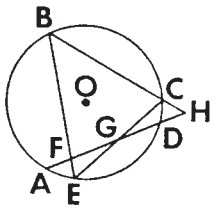
17. In triangle ABC, $\overline{CD}$ and $\overline{AE}$ are medians intersecting at O. $\overline{DE}$ is drawn. (*a*) Why is $\overline{DE}$ parallel to $\overline{AC}$? (*b*) Prove triangles ACO and EDO similar. (*c*) Write a proportion involving the measures of the sides of these triangles.

18. Prove that two isosceles triangles with congruent vertex angles are similar.

19. Given two isosceles triangles with a base angle of one congruent to a base angle of the other, prove that the triangles are similar.

20. Prove that the line segments which join the midpoints of the three sides of a given triangle form a triangle similar to the given triangle.

21. $\overline{HB}$ and $\overline{HA}$ are secants to circle O. Chords $\overline{EB}$ and $\overline{EC}$ intersect $\overline{AH}$ in F and G respectively. $m\overset{\frown}{AB}$: $m\overset{\frown}{BC}$: $m\overset{\frown}{CD} = 7{:}6{:}1$.



Ex. 21

a. Letting n equal the number of degrees in $\overset{\frown}{CD}$, find in terms of n the number of degrees in $\overset{\frown}{BC}$ and $\overset{\frown}{AB}$.

b. Prove that angle E is congruent to angle H.

c. Prove that triangle GEF is similar to triangle GHC.

d. If $CH = 20$, $EG = 14$, and $EF = 10$, find GH.

22. In isosceles trapezoid $ABCD$, $\overline{BC} \cong \overline{AD}$. If diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E, prove that $\triangle DEA \sim \triangle CEB$.

23. In circle O, M is the midpoint of $\overset{\frown}{RS}$ and a chord is drawn through M meeting chord $\overline{RS}$ at N and the circle at P. Chords $\overline{RM}$ and $\overline{RP}$ are drawn. Prove that $\triangle RPM \sim \triangle NRM$.

24. Two circles O and O' are tangent externally at P. Through P, a line is drawn which intersects circle O at S and circle O' at G. Through P, a second line is drawn which intersects circle O at R and circle O' at H. $\overline{SR}$ and $\overline{HG}$ are drawn. Prove that $\triangle SPR \sim \triangle HPG$. [*Hint:* Through P, draw the common tangent to circles O and O'.]

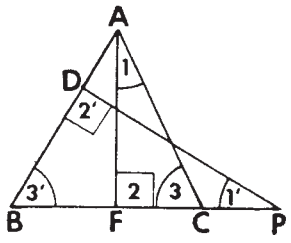
5. Using Similar Triangles to Prove Proportions Involving Line Segments

To prove that the measures of four line segments are in proportion, show that the line segments are corresponding sides in similar triangles. Try to find two triangles each of which has as sides two of the segments whose measures are mentioned in the proportion. Sometimes the numerators in the proportion are the measures of the sides of one triangle, and the denominators in the proportion are the measures of the sides of the other triangle. At other times, the terms of the first ratio in the proportion are the measures of the sides of one triangle, and the terms of the second ratio are the measures of the sides of the other triangle. When the triangles are selected, mark the angles of the triangles so that the angle opposite one segment involved in the proportion is named $\angle 1$, and the angle opposite the corresponding segment involved in the proportion is named $\angle 1'$. In a similar way, mark the angles opposite the second pair of corresponding segments involved in the proportion $\angle 2$ and $\angle 2'$. Mark the third pair of angles in the triangles $\angle 3$ and $\angle 3'$. Use any proper method to prove the triangles similar.

MODEL PROBLEM

In isosceles triangle ABC, $\overline{AB} \cong \overline{AC}$. $\overline{AF}$ is the altitude upon $\overline{BC}$. Through D, which is a point on $\overline{AB}$, a perpendicular to $\overline{AB}$ is drawn which meets $\overline{BC}$, extended if necessary, at P. Prove $FC : DB = AC : PB$.

Given: $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$.
$\overline{PD} \perp \overline{AB}$.
$\overline{AF} \perp \overline{BC}$.



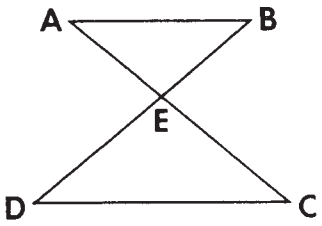
To prove: $\dfrac{FC}{DB} = \dfrac{AC}{PB}$

(FC and AC are sides of $\triangle FCA$; DB and PB are sides of $\triangle DBP$.)

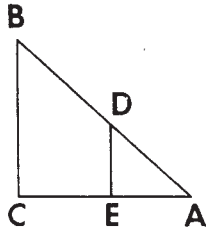
Plan: $\overline{FC}$ and $\overline{AC}$ are sides of $\triangle FCA$. $\overline{DB}$ and $\overline{PB}$ are sides of $\triangle DBP$. Mark the angles of $\triangle FCA$, $\angle 1$, $\angle 2$, and $\angle 3$. Mark the corresponding angles of $\triangle DBP$, $\angle 1'$, $\angle 2'$, and $\angle 3'$. Prove the triangles similar by proving two pairs of corresponding angles congruent.

Proof: *Statements*	*Reasons*
1. $\overline{AB} \cong \overline{AC}$.	1. Given.
2. $\angle 3 \cong \angle 3'$. (a. $\cong$ a.)	2. If two sides of a triangle are congruent, the angles opposite these sides are congruent.
3. $\overline{AF} \perp \overline{BC}$, $\overline{PD} \perp \overline{AB}$.	3. Given.
4. $\angle 2$ and $\angle 2'$ are right angles.	4. Perpendicular lines intersect forming right angles.
5. $\angle 2 \cong \angle 2'$. (a. $\cong$ a.)	5. All right angles are congruent.
6. $\triangle FCA \sim \triangle DBP$.	6. a.a. $\cong$ a.a.
7. $\frac{FC \text{ (opp. } \angle 1)}{DB \text{ (opp. } \angle 1')} = \frac{AC \text{ (opp. } \angle 2)}{PB \text{ (opp. } \angle 2')}$.	7. Corresponding sides of similar triangles are in proportion.

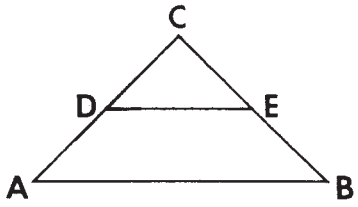
EXERCISES



Ex. 1



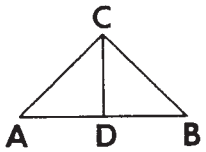
Ex. 2



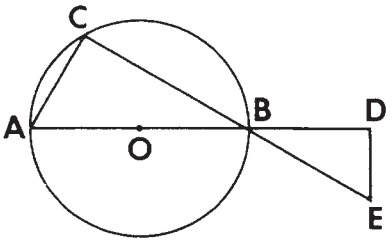
Ex. 3

1. $\overline{AB}$ is parallel to $\overline{DC}$. Prove: $\frac{AE}{CE} = \frac{BE}{DE}$.
2. In right $\triangle ABC$, $m\angle C = 90$ and $\overline{DE} \perp \overline{CA}$. Prove: $\frac{AD}{AB} = \frac{DE}{BC}$.
3. In $\triangle ABC$, D is the midpoint of $\overline{AC}$ and E is the midpoint of $\overline{BC}$. Prove: $CD{:}CA = DE{:}AB$.

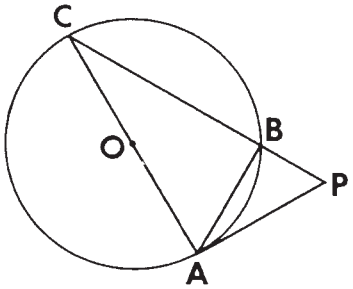
Exercises 4–6 on the next page refer to the following figures:



Ex. 4

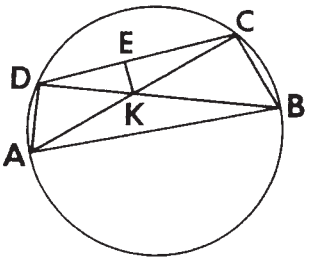


Ex. 5

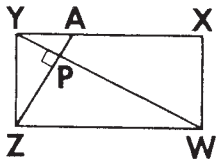


Ex. 6

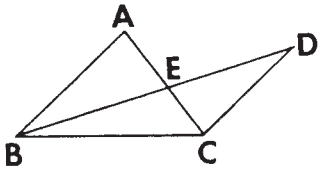
4. In right triangle ABC, $m\angle C = 90$ and $\overline{CD} \perp \overline{AB}$. Prove: $AB:AC = AC:AD$.
5. In circle O, $\overline{AB}$ is a diameter. $\overline{AB}$ is extended through B to D. $\overline{ED} \perp \overline{AD}$. $\overleftrightarrow{CBE}$ is a line. Prove: $AB:BC = EB:BD$.
6. In circle O, $\overline{PA}$ is a tangent and $\overline{PC}$ is a secant. Prove that AP is the mean proportional between PC and PB.
7. In triangle ABC, altitudes $\overline{CE}$ and $\overline{BD}$ intersect at F. Prove that $AB:AC = BD:CE$.
8. In triangle ABC, D is a point on $\overline{AB}$ and E is a point on $\overline{AC}$. $\overline{DE}$ is parallel to $\overline{BC}$. Prove that $AD:AB = DE:BC$.
9. Prove that the diagonals of a trapezoid divide each other proportionally.
10. In a circle, $\overline{AB}$ is a diameter and $\overline{BC}$ is a tangent. $\overline{AC}$ intersects the circle at D. Prove that $AD:AB = AB:AC$.
11. In triangle ABC, medians $\overline{BD}$ and $\overline{AE}$ intersect in F.
    a. Prove that $\overline{DE}$ is parallel to $\overline{AB}$.
    b. Prove that $ED:AB = EF:AF$.
    c. Prove that $DF:BF = EF:AF$.
    d. Find the value of the ratio $ED:AB$.
    e. Find the value of the ratio $DF:BF$.
    f. What fractional part of BD is BF?
    g. What fractional part of AE is FE?
12. In right triangle ABC, $\overline{CD}$ is the altitude to the hypotenuse $\overline{AB}$. Prove that BC is the mean proportional between AB and BD.
13. Triangle ABC is inscribed in circle O. $\overline{AF}$ is the altitude on $\overline{BC}$. $\overline{AD}$ is a diameter of the circle. Prove that $AC:AF = AD:AB$.
14. Prove that the measures of a pair of corresponding altitudes of similar triangles have the same ratio as the measures of a pair of corresponding sides.
15. *Prove:* In two similar triangles, the measures of the bisectors of a pair of corresponding angles have the same ratio as the measures of a pair of corresponding sides.
16. $ABCD$ is a parallelogram with side $\overline{BC}$ extended through C to any point E. $\overline{AE}$ is drawn intersecting $\overline{DC}$ in F. Prove that $CF:DF = CE:CB$.



Ex. 17



Ex. 18



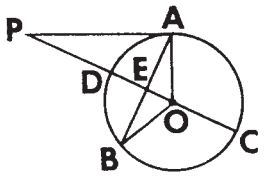
Ex. 19

17. $ABCD$ is a quadrilateral inscribed in a circle. $\overline{AB}$ is a diameter, and diagonals $\overline{AC}$ and $\overline{DB}$ intersect at K. $\overline{KE}$ is perpendicular to $\overline{DC}$.
Prove: $\dfrac{AB}{DK} = \dfrac{AC}{DE}$.

18. Given rectangle $WXYZ$ with A a point on $\overline{XY}$ such that $\overline{WY}$ intersects $\overline{ZA}$ at point P and $\overline{WY} \perp \overline{ZA}$.
Prove: (*a*) $\triangle WPZ \sim \triangle WZY$ (*b*) $\triangle WPZ \sim \triangle YPA$ (*c*) $YP{:}WZ = YA{:}WY$.

19. *Given:* $\angle ABE \cong \angle EBC$, $\overline{CD} \parallel \overline{AB}$.
Prove: (*a*) $\overline{BC} \cong \overline{CD}$ (*b*) $AB{:}CD = AE{:}CE$ (*c*) $AB{:}BC = AE{:}CE$.

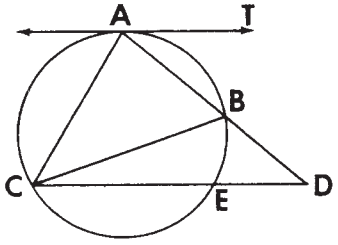
20. $\overline{PA}$ is a tangent to circle O at point A. Secant $\overline{PDC}$ passes through the center of the circle, O, and is perpendicular to chord $\overline{AB}$ at E. Radii $\overline{OA}$ and $\overline{OB}$ are drawn. Prove that $OP{:}OB = PA{:}BE$.



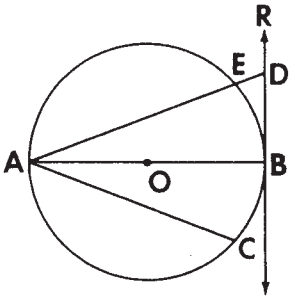
Ex. 20

21. In circle O, diameter $\overline{AB}$ is perpendicular to diameter $\overline{CD}$, and $\overline{AM}$ is any chord intersecting $\overline{CD}$ at P. Line segments $\overline{MB}$ and $\overline{BP}$ are drawn.
a. Prove that $AP = BP$.
b. Prove that $OB{:}MA = OP{:}BM$.

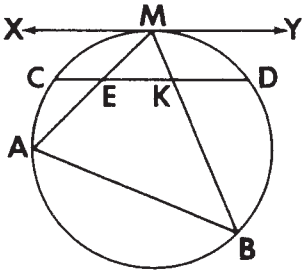
22. In circle O, $\overline{AB}$ and $\overline{AC}$ are congruent chords. A chord $\overline{AE}$ intersects chord $\overline{BC}$ in D. Prove that AB is the mean proportional between AE and AD.



Ex. 23



Ex. 24



Ex. 25

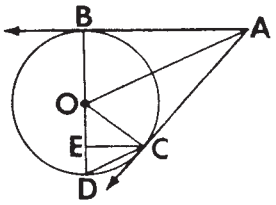
23. $\overleftrightarrow{AT}$ is tangent to the given circle. $\overline{CE}$ is a chord parallel to $\overleftrightarrow{AT}$, and B is any point on $\overset{\frown}{AE}$. $\overline{AC}$, $\overline{BC}$, and $\overline{AB}$ are drawn. $\overline{AB}$ and $\overline{CE}$ are extended to meet at D. Prove that AC is the mean proportional between AB and AD.

24. $\overline{AB}$ is a diameter of circle O, $\overleftrightarrow{RB}$ is a tangent at B, chords $\overline{AE}$ and $\overline{AC}$ are drawn on opposite sides of $\overline{AB}$ such that arc $\overset{\frown}{BE}$ is congruent to arc $\overset{\frown}{BC}$, and chord $\overline{AE}$ is extended to meet $\overleftrightarrow{RB}$ at D.
Prove: $AD{:}AB = AB{:}AC$.

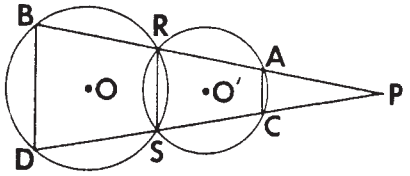
25. Triangle MAB is inscribed in a circle. $\overleftrightarrow{XY}$ is tangent to the circle at point

M. Chord $\overline{CD}$ is parallel to $\overleftrightarrow{XY}$ and intersects $\overline{MA}$ and $\overline{MB}$ at points E and K respectively.

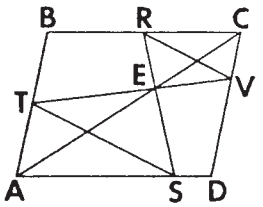
Prove: $\dfrac{ME}{MB} = \dfrac{MK}{MA}$.



Ex. 26



Ex. 27



Ex. 28

26. $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are tangents to circle O at points B and C respectively. From C, a line is drawn perpendicular to diameter $\overline{BD}$ and intersecting $\overline{BD}$ at E. Line segments $\overline{AO}$, $\overline{OC}$, and $\overline{DC}$ are drawn. Prove that: (*a*) $\overline{AO}$ bisects arc $\overset{\frown}{BC}$. (*b*) $\angle BOA \cong \angle EDC$. (*c*) $AB{:}CE = BO{:}ED$.

27. Circles O and O' intersect at points R and S. Secants from point P, outside circles O and O', through R and S intersect the circles at points A and B, and C and D respectively. Chords $\overline{AC}$, $\overline{RS}$, and $\overline{BD}$ are drawn. (*a*) If the number of degrees contained in $\overset{\frown}{RBD}$ is represented by $2x$, express in terms of x the number of degrees contained in $\angle DSR$, $\angle DBR$, $\angle RSC$, and $\angle CAR$. (*b*) Prove that $PA{:}PB = PC{:}PD$.

28. Given parallelogram $ABCD$ with E a point on diagonal $\overline{AC}$. Lines through E intersect $\overline{BC}$ at R, $\overline{AD}$ at S, $\overline{AB}$ at T, and $\overline{CD}$ at V. *Prove:* (*a*) $AE{:}CE = TE{:}VE$. (*b*) $AE{:}CE = SE{:}RE$. (*c*) Triangle TES is similar to triangle VER. (*d*) $\overline{TS}$ is parallel to $\overline{RV}$.

6. Proving That Products Involving Line Segments Are Equal

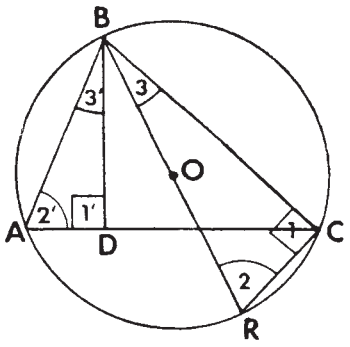
To prove that the product of the measures of two line segments is equal to the product of the measures of two other line segments:

1. Form a proportion in which the measures of the four line segments appear. Do this by using the theorem "If the product of two numbers (not zero) is equal to the product of two other numbers (not zero), either pair of numbers may be made the means and the other pair the extremes in a proportion."
2. Prove this proportion by the procedure outlined on page 294.
3. Use the theorem "In a proportion, the product of the means is equal to the product of the extremes."

MODEL PROBLEM

Triangle ABC is inscribed in a circle. $\overline{BD}$ is the altitude on $\overline{AC}$, and $\overline{BR}$ is a diameter. Prove that $BA \times BC = BR \times BD$.

Given: $\triangle ABC$ is inscribed in circle O.
$\overline{BD} \perp \overline{AC}$.
$\overline{BR}$ is a diameter.



To prove: $BA \times BC = BR \times BD$.

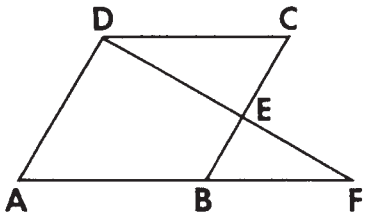
$$\left[\text{If } \frac{BR}{BA} = \frac{BC}{BD}\right] \quad \begin{matrix} \triangle BRC \\ \triangle BAD \end{matrix}$$

Plan: We can prove that $BA \times BC = BR \times BD$ if we can prove the proportion which we get when we make BR and BD the extremes and BA and BC the means. $BR{:}BA = BC{:}BD$. Name the pairs of corresponding angles 1 and 1', 2 and 2', 3 and 3'. We can prove this proportion by proving $\triangle BRC \sim \triangle BAD$ by a.a. $\cong$ a.a.

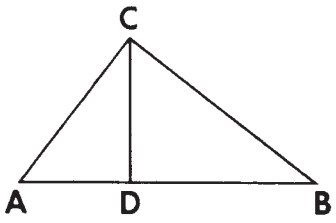
Proof: *Statements*	*Reasons*
1. Draw $\overline{RC}$ to form $\triangle BRC$.	1. One and only one straight line may be drawn between two points.
2. $\overline{BD} \perp \overline{AC}$, $\overline{BR}$ is a diameter.	2. Given.
3. $\angle 1$ is a right angle.	3. An angle inscribed in a semicircle is a right angle.
4. $\angle 1'$ is a right angle.	4. Perpendicular lines intersect forming right angles.
5. $\angle 1 \cong \angle 1'$ (a. $\cong$ a.)	5. All right angles are congruent.
6. $m\angle 2 = \frac{1}{2}m\overset{\frown}{BC}$, $m\angle 2' = \frac{1}{2}m\overset{\frown}{BC}$.	6. The measure of an inscribed angle is equal to one-half the measure of its intercepted arc.
7. $m\angle 2 = m\angle 2'$.	7. Transitive property of equality.
8. $\angle 2 \cong \angle 2'$. (a. $\cong$ a.)	8. Definition of congruent angles.
9. $\triangle BRC \sim \triangle BAD$.	9. a.a. $\cong$ a.a
10. $\frac{BR \text{ (opp. } \angle 1)}{BA \text{ (opp. } \angle 1')} = \frac{BC \text{ (opp. } \angle 2)}{BD \text{ (opp. } \angle 2')}$.	10. Corresponding sides of similar triangles are in proportion.
11. $BA \times BC = BR \times BD$.	11. In a proportion, the product of the means is equal to the product of the extremes.

EXERCISES

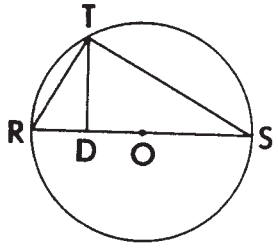
Exercises 1–3 refer to the figures below.



Ex. 1

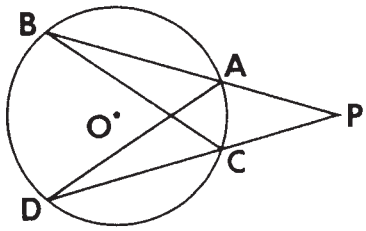


Ex. 2



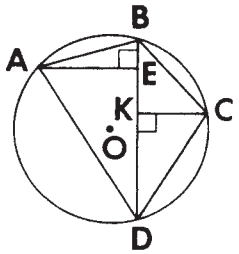
Ex. 3

1. If $ABCD$ is a parallelogram, prove that $DE \times BE = FE \times CE$.
2. In right triangle ABC, $\angle C$ is a right angle and $\overline{CD} \perp \overline{AB}$. Prove that $AD \times CB = DC \times AC$.
3. In circle O, $\overline{RS}$ is a diameter, T is a point on $\overset{\frown}{RS}$, and $\overline{TD} \perp \overline{RS}$. Prove that $RS \times RD = (RT)^2$. $[(RT)^2 = RT \times RT]$
4. In circle O, $\overline{PB}$ and $\overline{PD}$ are secants. Prove that $PB \times PA = PD \times PC$.
5. If $\overline{AB}$ is parallel to $\overline{CD}$, and $\overline{AC}$ and $\overline{BD}$ intersect at E, prove that $DE \times AE = BE \times CE$.
6. In triangle ABC, altitudes $\overline{CD}$ and $\overline{AE}$ intersect at F. Prove that $FD \times CF = FE \times AF$.

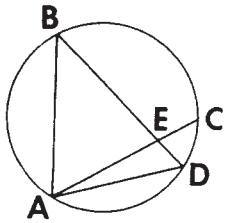


Ex. 4

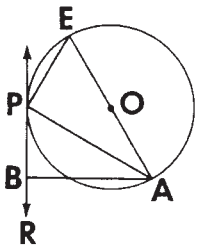
7. In right triangle ABC, angle C is the right angle. From D, a point on a leg $\overline{AC}$, $\overline{DE}$ is drawn perpendicular to $\overline{AB}$. Prove that $AC \times AD = AE \times AB$.
8. In a circle whose center is O, $\overline{AB}$ is a diameter and $\overline{AE}$ a chord. From any point C on chord $\overline{AE}$, $\overline{CD}$ is drawn perpendicular to $\overline{AB}$. Prove that $AB \times AD = AC \times AE$.
9. Triangle ABC is inscribed in a circle. The bisector of angle C intersects side $\overline{AB}$ at D and $\overset{\frown}{AB}$ at E. *Prove:* $AC \times CB = CD \times EC$.
10. If $\overline{AB}$ is the diameter of a circle, $\overline{AD}$ a chord, and $\overline{DC} \perp \overline{AB}$, prove that $(AD)^2 = AB \times AC$.



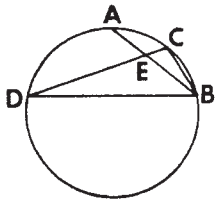
Ex. 11



Ex. 12

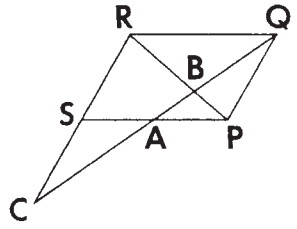


Ex. 13



Ex. 14

11. Quadrilateral $ABCD$ is inscribed in circle O. Perpendiculars are drawn to diagonal $\overline{BD}$ from A and C, meeting $\overline{BD}$ at E and K respectively. If $\overset{\frown}{AB}$ is congruent to $\overset{\frown}{BC}$, prove that $ED \times CD = AD \times KD$.
12. B is the midpoint of major arc $\overset{\frown}{AC}$. Chords $\overline{BD}$ and $\overline{AC}$ intersect at E. Chords $\overline{AD}$ and $\overline{AB}$ are drawn. *Prove:* $BD \times BE = (BA)^2$.
13. $\overleftrightarrow{PR}$ is a tangent, $\overline{PA}$ and $\overline{PE}$ are chords, and $\overline{AE}$ is a diameter of circle O. $\overline{AB}$ is perpendicular to $\overleftrightarrow{PR}$. *Prove:* $AB \times AE = (AP)^2$.
14. C is the midpoint of arc $\overset{\frown}{AB}$. Chords $\overline{AB}$ and $\overline{CD}$ intersect in E, and chords $\overline{CB}$ and $\overline{BD}$ are drawn. Prove that $CD \times CE = (CB)^2$.
15. In isosceles triangle ABC, $\overline{BA} \cong \overline{AC}$. The bisectors of $\angle B$ and $\angle C$ intersect the congruent sides in D and E respectively. Prove that $AE \times AC = AB \times AD$.
16. In triangle ABC, $\overline{BD}$ bisects angle B and intersects $\overline{AC}$ at D. Through C, a line parallel to $\overline{BD}$ is drawn to meet $\overline{AB}$ extended in E. *Prove:*
    a. Triangle EBC is isosceles.
    b. $AD \times CB = AB \times CD$.
17. In parallelogram $PQRS$, $\overline{QC}$ intersects diagonal $\overline{RP}$ at B, side $\overline{SP}$ at A, and extended side $\overline{RS}$ at C.
    a. Prove that $\triangle APB \sim \triangle QRB$.
    b. Prove that $\triangle QBP \sim \triangle CBR$.
    c. Prove that $(QB)^2 = AB \times BC$.



Ex. 17

7. Using Proportions Involving Corresponding Line Segments in Similar Triangles

We have already learned from the definition of similar polygons that if two polygons are similar, their corresponding sides are in proportion. Since triangles are polygons, it follows that:

If two triangles are similar, their corresponding sides are in proportion.

If $\triangle ABC \sim \triangle A'B'C'$ (Fig. 6–16), $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$. The ratio of the measures of any pair of corresponding sides of the similar triangles, $\frac{a}{a'}$ or $\frac{b}{b'}$ or $\frac{c}{c'}$, is called the *ratio of similitude* of the similar triangles.

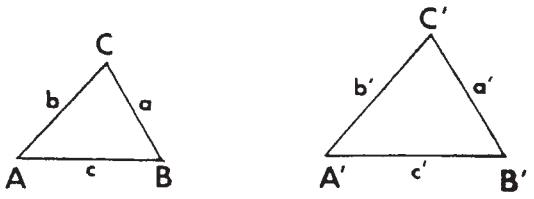


Fig. 6–16

Theorem 91. **If two triangles are similar, the measures of corresponding altitudes have the same ratio as the measures of any two corresponding sides.**

In Fig. 6–17, if $\triangle ABC \sim \triangle A'B'C'$, h and h' represent the measures of two corresponding altitudes, and a and a' represent the measures of two corresponding sides, then $\frac{h}{h'} = \frac{a}{a'}$.

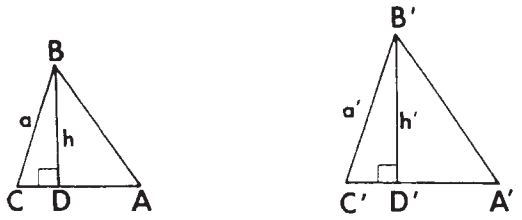


Fig. 6–17

In theorem 91, if the expression "corresponding altitudes" is replaced by "corresponding medians" or by "corresponding angle bisectors," the resulting theorems would also be true. We therefore have the following corollary to theorem 91:

Corollary T91–1. **In two similar triangles, the measures of any two corresponding line segments have the same ratio as the measures of any pair of corresponding sides.**

If $\triangle ABC \sim \triangle A'B'C'$, then:

$$\frac{\text{measure of any line seg. in } \triangle ABC}{\text{measure of corr. line seg. in } \triangle A'B'C'} = \frac{\text{measure of any side in } \triangle ABC}{\text{measure of corr. side in } \triangle A'B'C'}$$

Theorem 92. **The perimeters of two similar triangles have the same ratio as the measures of any pair of corresponding sides.**

In Fig. 6–18, if $\triangle ABC \sim \triangle A'B'C'$, p and p' represent their perimeters, and a and a' represent the measures of a pair of corresponding sides, then $\frac{p}{p'} = \frac{a}{a'}$.

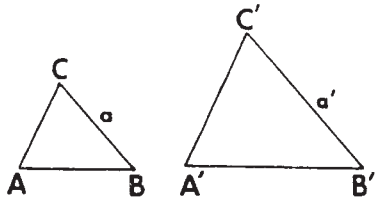


Fig. 6–18

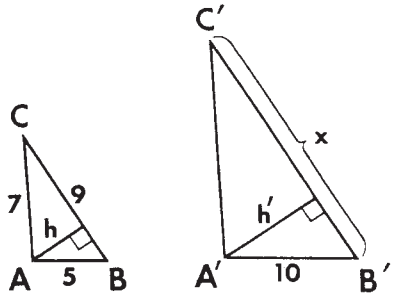
MODEL PROBLEMS

1. The sides of triangle ABC measure 5, 7, and 9. The shortest side of a similar triangle $A'B'C'$ measures 10.
   a. Find the measure of the longest side of triangle $A'B'C'$.
   b. Find the ratio of the measures of a pair of corresponding altitudes in triangles ABC and $A'B'C'$.
   c. Find the perimeter of triangle $A'B'C'$.

   Solution:

   a. 1. The longest side of $\triangle A'B'C'$ is $\overline{B'C'}$ because it corresponds to $\overline{BC}$, the longest side in $\triangle ABC$.

2. Since $\triangle A'B'C' \sim \triangle ABC$, then
$\frac{AB}{A'B'} = \frac{BC}{B'C'}$ Let x = the length of $B'C'$.



3. $\frac{5}{10} = \frac{9}{x}$

4. $5x = 90$

5. $x = 18$

Answer: The longest side of $\triangle A'B'C'$ measures 18.

b. Since $\triangle ABC \sim \triangle A'B'C'$, then $\frac{h}{h'} = \frac{AB}{A'B'} = \frac{5}{10} = \frac{1}{2}$.

Answer: $h:h' = 1:2$

c. 1. Since $\triangle ABC \sim \triangle A'B'C'$, then
$\frac{p}{p'} = \frac{s}{s'}$ Perimeter of $\triangle ABC = 5 + 7 + 9 = 21$.
Let y = the perimeter of $\triangle A'B'C'$.

2. $\frac{21}{y} = \frac{5}{10}$

3. $5y = 210$

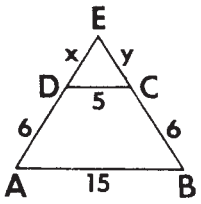
4. $y = 42$

Answer: The perimeter of $\triangle A'B'C' = 42$.

2. In an isosceles trapezoid, the length of the lower base is 15, the length of the upper base is 5, and the length of each congruent side is 6. How many units must each nonparallel side be extended to form a triangle?

Solution:

1. Since $\overline{DC} \parallel \overline{AB}$, then $\triangle DEC \sim \triangle AEB$ and their corresponding sides are in proportion.

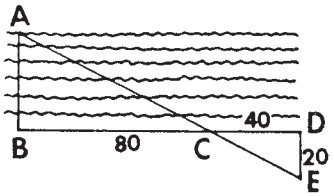


2.	$\frac{ED}{EA} = \frac{DC}{AB}$	2.	$\frac{EC}{EB} = \frac{DC}{AB}$
	Let $ED = x$. Hence,		Let $EC = y$. Hence,
	$EA = x + 6$		$EB = y + 6$
3.	$\frac{x}{x+6} = \frac{5}{15}$	3.	$\frac{y}{y+6} = \frac{5}{15}$
4.	$15x = 5x + 30$	4.	$15y = 5y + 30$
5.	$10x = 30$	5.	$10y = 30$
6.	$x = 3$	6.	$y = 3$

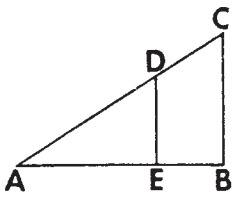
Answer: 3.

EXERCISES

1. The lengths of the sides of a triangle are 8, 10, and 12. If the length of the shortest side of a similar triangle is 6, find the length of its longest side.
2. Triangle DEF is similar to triangle $D'E'F'$. $\angle D$ corresponds to $\angle D'$ and $\angle E$ corresponds to $\angle E'$. If $DE = 2x + 2$, $DF = 5x - 7$, $D'E' = 2$, and $D'F' = 3$, find DE and DF.
3. The ratio of similitude in two similar triangles is 3:1. If a side in the larger triangle measures 30, find the measure of the corresponding side in the smaller triangle.
4. A vertical pole 10 ft. high casts a shadow 8 ft. long, and at the same time a nearby tree casts a shadow 40 ft. long. What is the height of the tree?



Ex. 5



Ex. 6

5. In the figure, $\overline{AB}$ represents the width of a river. $\overline{AE}$ and $\overline{BD}$ intersect at C. $\overline{AB} \perp \overline{BD}$ and $\overline{ED} \perp \overline{BD}$ Find AB, the width of the river.
6. $\overline{DE} \perp \overline{AB}$, and $\overline{CB} \perp \overline{AB}$. If $CB = 40$, $DE = 30$, and $EB = 20$, find AE.
7. A boy 5 ft. tall stands on level ground 6 ft. from a point P which is directly below a light. (*a*) If the boy's shadow is 3 ft. long, find the height of the light above the ground. (*b*) If the boy takes a position 2 ft. nearer P, find the length of his shadow.
8. The sides of a triangle are 10, 12, 15. A line segment whose length is 5 is parallel to the longest side of the triangle and has its endpoints on the other two sides of the triangle. Find the shorter segment of side 10.
9. In triangle ABC, a line parallel to $\overline{AB}$ intersects $\overline{AC}$ at D and $\overline{CB}$ at E. If $CD = 6$, $DA = 12$, and AB is 8 less than 4 times DE, find DE and AB.
10. The bases of an isosceles trapezoid are 5 and 10, and each of the non-parallel sides is 4. How many units must each of the nonparallel sides be extended to form a triangle?
11. The bases of a trapezoid are 10 and 15 and the nonparallel sides are 4 and 5. How many units must each of the nonparallel sides be extended to form a triangle?
12. If the lengths of the sides of two similar triangles are in the ratio 5:1, find the ratio of the lengths of a pair of corresponding altitudes.
13. The lengths of two corresponding sides of two similar triangles are 8

and 12. If an altitude of the smaller triangle is 6, find the length of the corresponding altitude of the larger triangle.

14. The lengths of two corresponding altitudes in two similar triangles are 18 and 27. If the length of a side in the larger triangle exceeds the length of the corresponding side of the smaller triangle by 12, find the lengths of these two sides of the triangles.

15. A side of a triangle is 20 inches and the altitude drawn to this side is 12 inches. Find the length of a line segment drawn parallel to the given side, through a point inside the triangle which is 3 inches from the vertex, and which terminates in the other two sides.

16. The bases of a trapezoid are 10 and 40 and the altitude is 6. The nonparallel sides are extended until they intersect. Find the distance between the point of intersection and the shorter base of the trapezoid.

17. The ratio of similitude in two similar triangles is 4:3. If the length of a median in the first triangle is 12, find the length of the corresponding median in the second triangle.

18. In two similar triangles, the ratio of the lengths of two corresponding angle bisectors is 5:3. If the length of a side of the smaller triangle is 21, find the length of the corresponding side of the larger triangle.

19. The ratio of the lengths of the corresponding sides of two similar triangles is 7:4. Find the ratio of the perimeters of the triangles.

20. The sides of a triangle are 7, 9, and 11. Find the perimeter of a similar triangle in which the side corresponding to 7 in the first triangle is 21.

21. Corresponding altitudes of two similar triangles are 9 and 6. If the perimeter of the larger triangle is 24, what is the perimeter of the smaller triangle?

22. In two similar triangles, the ratio of the lengths of two corresponding sides is 5:8. If the perimeter of the larger triangle is 10 less than twice the perimeter of the smaller triangle, find the perimeter of each triangle.

8. Using Proportions Involving Corresponding Line Segments in Similar Polygons

We have already learned from the definition of similar polygons that if polygon $ABCDE \sim$ polygon $A'B'C'D'E'$ (Fig. 6–19), then their corresponding sides are in proportion, or

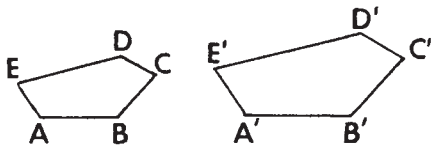


Fig. 6–19

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}$$

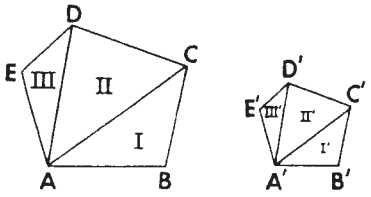


Fig. 6–20

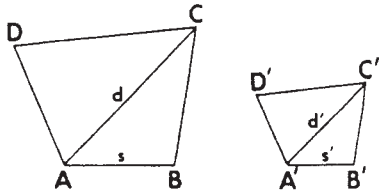


Fig. 6–21

Theorem 93. **Similar polygons may be divided, by drawing the diagonals from a pair of corresponding vertices, into triangles so that the corresponding triangles are similar.**

In Fig. 6–20, if polygon $ABCDE \sim$ polygon $A'B'C'D'E'$, then:

$$\triangle I \sim \triangle I',\ \triangle II \sim \triangle II',\ \triangle III \sim \triangle III'$$

Corollary T93–1. **If two polygons are similar, the ratio of the lengths of two corresponding diagonals is equal to the ratio of the lengths of any two corresponding sides.**

In Fig. 6–21, if polygon $ABCD \sim$ polygon $A'B'C'D'$, then $\frac{d}{d'} = \frac{s}{s'}$.

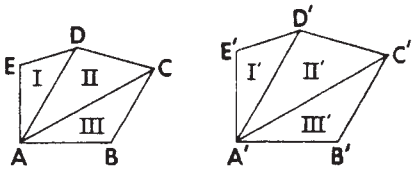


Fig. 6–22

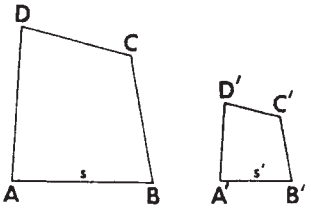


Fig. 6–23

Theorem 94. **Two polygons are similar if they are composed of the same number of triangles similar each to each and similarly placed.**

In Fig. 6–22, if $\triangle I \sim \triangle I'$, $\triangle II \sim \triangle II'$, $\triangle III \sim \triangle III'$, and all pairs of $\triangle$'s are similarly placed, then polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

Theorem 95. **The perimeters of two similar polygons have the same ratio as the measures of any pair of corresponding sides.**

In Fig. 6–23, if polygon $ABCD \sim$ polygon $A'B'C'D'$, and p and p' represent their perimeters, and s and s' represent the measures of a pair of corresponding sides, then $\frac{p}{p'} = \frac{s}{s'}$.

Corollary T95–1. **The perimeters of two similar polygons have the same ratio as the measures of any pair of corresponding line segments.**

If p and p' represent the perimeters of two similar polygons and l and l' represent the measures of any pair of corresponding line segments of these polygons, then $\frac{p}{p'} = \frac{l}{l'}$.

MODEL PROBLEM

The lengths of two corresponding sides of two similar polygons are 4 and 7. If the perimeter of the smaller polygon is 20, find the perimeter of the larger polygon.

Solution:

1. Since the polygons are similar, then

$$\frac{p}{p'} = \frac{s}{s'} \qquad s = 4,\ s' = 7,\ p = 20.$$
Let $p' = x$.

2. $\frac{20}{x} = \frac{4}{7}$

3. $4x = 140$

4. $x = 35$

Answer: The perimeter of the larger polygon is 35.

EXERCISES

1. The lengths of two corresponding sides of two similar polygons are 5 and 20. Find the ratio of similitude of the two polygons.
2. Two corresponding diagonals of two similar polygons are 4 and 8. Find the ratio of similitude of the two polygons.
3. The ratio of similitude of two similar polygons is 3:4, and a diagonal of the larger polygon is 20. Find the length of the corresponding diagonal in the smaller polygon.
4. Two corresponding sides of two similar polygons are 8 and 12. Find the ratio of their perimeters.
5. Two corresponding sides of two similar polygons are 9 and 12. If the perimeter of the larger polygon is 72, find the perimeter of the smaller polygon.
6. The perimeter and side of a polygon are 30 and 6 respectively. Find the length of the corresponding side of a similar polygon whose perimeter is 20.
7. Two corresponding sides of two similar polygons are 14 and 7. If the perimeter of the smaller polygon is 23 less than the perimeter of the larger polygon, find the perimeter of the larger polygon.
8. Two corresponding diagonals of two similar polygons are in the ratio 3 to 1. If the perimeter of the larger polygon exceeds the perimeter of the smaller polygon by 48, find the perimeter of the smaller polygon.

9. Two Chords Intersecting Inside a Circle

Theorem 96. **If two chords intersect inside a circle, the product of the measures of the segments of one chord is equal to the product of the measures of the segments of the other.**

If chords $\overline{AB}$ and $\overline{CD}$ intersect at E (Fig. 6–24), the product of the measures of the segments of chord $\overline{AB}$, $AE \times EB$, is equal to the product of the measures of the segments of chord $\overline{CD}$, $CE \times ED$, or:

$$AE \times EB = CE \times ED$$

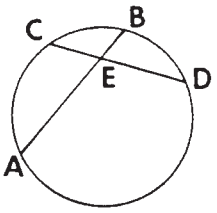


Fig. 6–24

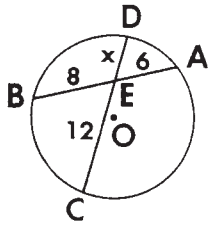
MODEL PROBLEMS

1. In circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect at E. If $AE = 6$, $EB = 8$, and $CE = 12$, find ED.

Solution:

1. $AE \times EB = CE \times ED$ Let $ED = x$.
2. $6 \times 8 = 12x$
3. $48 = 12x$
4. $4 = x$

Answer: $ED = 4$.

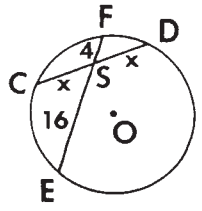


2. In circle O, chord $\overline{CD}$ is bisected at S by chord $\overline{EF}$. If $ES = 16$ and $SF = 4$, find the length of chord $\overline{CD}$.

Solution:

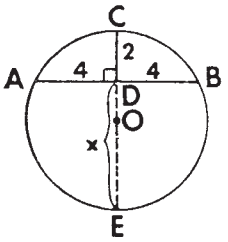
1. Since $\overline{EF}$ bisects $\overline{CD}$, $\overline{CS} \cong \overline{SD}$, and $CS = SD$.
2. $CS \times SD = ES \times SF$ Let x = the length of $\overline{CS}$ and the length of $\overline{SD}$.
3. $(x)(x) = (16)(4)$
4. $x^2 = 64$
5. $x = 8$
6. $CD = 8 + 8 = 16$

Answer: $CD = 16$.



3. In circle O, the length of the chord of a minor arc is 8. The height of the arc (the length of the line segment joining the midpoint of the arc to the midpoint of its chord) is 2. Find the length of the radius of the circle.

Solution:



1. Since $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$, when $\overline{CD}$ is extended it will pass through the center of the circle, O.
2. Since $AB = 8$ and $AD = DB$, $AD = 4$ and $DB = 4$.
3. $AD \times DB = CD \times DE$ Let $DE = x$.
4. $4 \times 4 = 2x$
5. $16 = 2x$
6. $x = 8$
7. The length of diameter $\overline{CE} = 2 + x = 2 + 8 = 10$.
8. The length of radius $\overline{OE} = \frac{1}{2}CE = \frac{1}{2}(10) = 5$.

Answer: The length of the radius is 5.

EXERCISES

1. Two chords intersect inside a circle. The segments of one chord measure 8 and 3. If one segment of the second chord measures 6, then the measure of the other segment is _____.
2. Chords $\overline{AB}$ and $\overline{CD}$ intersect inside a circle at point E. $AE = 4$, $CE = 3$, and $ED = 8$. Find EB.
3. A diameter of a circle is perpendicular to a chord whose length is 14 inches. If the length of the shorter segment of the diameter is 4 inches, find the length of the longer segment of the diameter.
4. Chords $\overline{AB}$ and $\overline{CD}$ intersect inside circle O at point E. If $AE = 9\frac{3}{5}$, $EB = 1\frac{1}{4}$, and $DE = 3$, find EC.
5. Chords $\overline{AB}$ and $\overline{CE}$ intersect inside a circle at D. If $AD = a$, $DB = b$, and $CD = c$, express DE in terms of a, b, and c.
6. In circle O, chord $\overline{AB}$ bisects chord $\overline{CD}$ at E. If $AE = 16$ and $EB = 4$, find the length of $\overline{CE}$.
7. Chords $\overline{AB}$ and $\overline{CD}$ intersect inside a circle at point E. If $CD = 9$, $ED = 8$, and AE is twice EB, find EB.
8. Chords $\overline{AB}$ and $\overline{CD}$ intersect inside a circle at point E. If $AE = 25$, $EB = 8$, and the ratio of $CE:ED$ is 1:2, find CD.

9. A chord of a circle is perpendicular to a diameter of the circle and divides the diameter into segments which are 2 inches and 18 inches in length. Find the length of the chord.
10. Point P is a distance of 6 from the center of a circle whose radius is 10; the product of the lengths of the segments of any chord drawn through P is _____.
11. Point P is 3 inches from the center of a circle whose radius is 5 inches. (*a*) Find the length of the shortest chord that can be drawn through P. (*b*) Find the length of the longest chord which can be drawn through P.
12. In a circle, chord $\overline{AB}$, which is 12 in. in length, is drawn. The height of minor arc $\overset{\frown}{AB}$ is 3 in. Find the length of the radius of the circle.
13. In a circle, chord $\overline{CD}$, which is 16 in. in length, is drawn. The center of minor arc $\overset{\frown}{CD}$ is joined to the midpoint of chord $\overline{CD}$ by a line segment 4 in. in length. Find the length of the radius of the circle.
14. Chords $\overline{AB}$ and $\overline{CD}$ intersect inside a circle at point E. $\overline{AE}$ measures 4 in., $\overline{DE}$ measures 3 in., and $\overline{EB}$ measures 4 inches shorter than $\overline{CE}$. (*a*) If the length of $\overline{CE}$ is represented by x, write an equation that can be used to find x. (*b*) Find the length of the shorter chord.
15. In a circle, chords $\overline{AB}$ and $\overline{CD}$ intersect in E. If $CE = 6$, $ED = 4$, $AE = x + 3$, and $EB = x - 2$, find AE, EB, and AB.
16. In a circle, chords $\overline{AB}$ and $\overline{CD}$ intersect at E. $\overline{DE}$ measures 12 in., $\overline{CE}$ measures 8 in., and AE exceeds BE by 20. (*a*) If the length of $\overline{BE}$ is represented by x, write an equation which can be used to find x. (*b*) Solve this equation for x. (*c*) Find the length of chord $\overline{AB}$.
17. In a given circle, chords $\overline{AB}$ and $\overline{CD}$ are drawn so that chord $\overline{AB}$ bisects chord $\overline{CD}$. Prove that the measure of each segment of chord $\overline{CD}$ is the mean proportional between the measures of the segments of chord $\overline{AB}$.
18. Prove that the product of the measures of the segments of the chords drawn through a point which is b inches from the center of a circle whose radius measures r inches long is constant.

10. Secants and Tangents Drawn to a Circle From an Outside Point

Definition. **The *distance from a point to a circle* is the shortest distance from the given point to a point of intersection of the circle with the line which passes through the given point and the center of the circle.**

In Fig. 6–25, if point P is outside circle O, and if when $\overline{PO}$ is drawn it intersects the circle at B, then the distance from P to B is the distance from point P to circle O.

Similarly, if point P' is inside circle O, the distance from P' to B' is the distance from point P' to circle O.

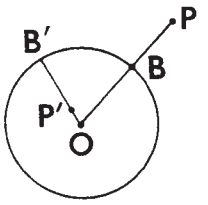


Fig. 6–25

If $\overleftrightarrow{PB}$ is a secant to circle O from outside point P (Fig. 6–26), then we will also refer to $\overline{PB}$ as a secant to circle O from point P. Then PB, which is the distance from P to the farther point of intersection B, is called the *length of the secant*. PA, which is the distance from P to the nearer point of intersection A, is called the *length of the external segment of the secant*. AB, which is the distance between the two points of intersection A and B, is called the *length of the internal segment of the secant*.

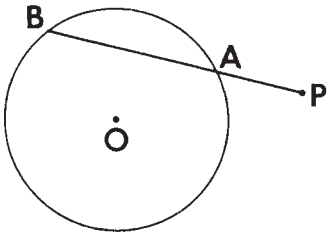


Fig. 6–26

Theorem 97. **If from a point outside a circle two secants are drawn to the circle, the product of the length of one secant and the length of its external segment is equal to the product of the length of the other secant and the length of its external segment.**

If $\overline{PB}$ and $\overline{PD}$ are secants to the circle (Fig. 6–27), the product of the length of secant $\overline{PB}$ and the length of its external segment $\overline{PA}$, $PB \times PA$, is equal to the product of the length of secant $\overline{PD}$ and the length of its external segment $\overline{PC}$, $PD \times PC$, or $PB \times PA = PD \times PC$.

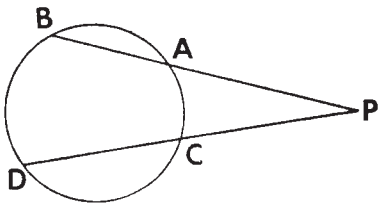


Fig. 6–27

Theorem 98. **If from a point outside a circle a tangent and a secant are drawn to the circle, the length of the tangent is the geometric mean between the length of the secant and the length of its external segment.**

If tangent $\overline{PA}$ and secant $\overline{PC}$ are drawn to the circle (Fig. 6–28), then the length of tangent $\overline{PA}$ is the geometric mean between the length of secant $\overline{PC}$ and the length of its external segment $\overline{PB}$, or $\dfrac{PB}{PA} = \dfrac{PA}{PC}$, or $(PA)^2 = PB \times PC$.

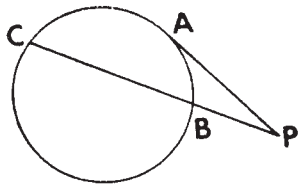
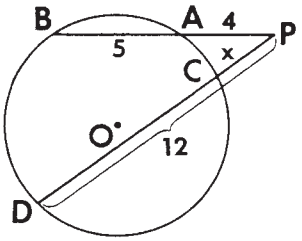


Fig. 6–28

MODEL PROBLEMS

1. $\overline{PB}$ and $\overline{PD}$, which are secants drawn to circle O, intersect the circle in points A and C respectively. If $PA = 4$, $AB = 5$, and $PD = 12$, find PC.

Solution:

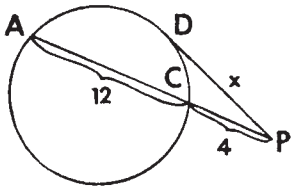


1. $PB \times PA = PD \times PC$ Let $x = PC$.
2. $9 \times 4 = 12x$
3. $36 = 12x$
4. $3 = x$

Answer: $PC = 3$.

2. From a point outside a circle, a tangent and a secant are drawn to the circle. The point at which the secant intersects the circle divides the secant into an external segment of length 4 and an internal segment of length 12. Find the length of the tangent.

Solution:



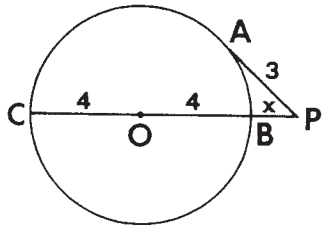
1. $PA:PD = PD:PC$ Let $x = PD$, the length of the tangent.
2. $16:x = x:4$
3. $x^2 = 64$
4. $x = 8$

Answer: $PD = 8$.

3. The length of a radius of circle O is 4. From a point P outside circle O, tangent $\overline{PA}$ is drawn. If the length of $\overline{PA}$ is 3, find the distance from P to the circle.

Solution:

1. PB is the distance from P to the circle.
2. $PC:PA = PA:PB$ Let $x = PB$.
3. $(x + 8):3 = 3:x$ Then $x + 8 = PC$.
4. $x(x + 8) = 3 \times 3$
5. $x^2 + 8x = 9$
6. $x^2 + 8x - 9 = 0$
7. $(x - 1)(x + 9) = 0$
8. $x - 1 = 0 \mid x + 9 = 0$
9. $x = 1 \mid x = -9$ Reject the negative value.

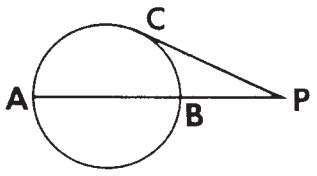


Answer: $PB = 1$.

EXERCISES

1. Two secants are drawn to a circle from an outside point. The length of one secant is 20 and the length of its external segment is 5. If the length of the other secant is 25, find the length of its external segment.
2. Two secants are drawn to a circle from an outside point. The external segment of the first secant is 4 and its internal segment is 8. Find the length of the second secant if its external segment is 6.
3. From point P outside circle O, tangent $\overline{PD}$ and secant $\overline{PA}$ are drawn. Secant $\overline{PA}$ intersects the circle at C. If $PD = 8$ and $PC = 4$, find the length of secant $\overline{PA}$.
4. A tangent and a secant are drawn to a circle from a point outside the circle. If the tangent is 14 in. long and the secant is 28 in. long, the length of the external segment of the secant is _____ in.
5. If the length of a secant to a circle from an external point is 9 and the length of its external segment is 4, the length of the tangent from that point is _____.
6. A tangent and a secant are drawn to a circle from the same outside point. If the length of the external segment of the secant is 3 and the length of the internal segment is 7, find the length of the tangent.
7. A tangent and a secant are drawn to a circle from the same external point. If the length of the tangent is 10 and the length of the external segment of the secant is 4, find the length of the internal segment of the secant.
8. From point P outside a circle, tangent $\overline{PD}$ and secant $\overline{PA}$ are drawn. Secant $\overline{PA}$ intersects the circle at C. $PD = 8$ and the length of the secant is four times the length of its external segment. Find the length of the secant.
9. From point P outside a circle, tangent $\overline{PD}$ and secant $\overline{PA}$ are drawn. Secant $\overline{PA}$ intersects the circle at C. If $PD = a$ and $PC = b$, express PA in terms of a and b.
10. The diameter of a circle is 15 inches. If this diameter is extended 5 inches beyond the circle to point A, find the number of inches in the length of a tangent to the circle from point A.

11. In the circle shown, $\overline{AB}$ is a diameter, $\overline{PC}$ is a tangent, and $\overline{PBA}$ is a secant. If $AP = 9$ and $CP = 6$, find the number of units in the diameter of the circle.



Ex. 11

12. The length of a tangent drawn from a point 3 inches from a circle the length of whose radius is 12 inches is _____ inches.

13. Point P is 15 in. from the center of a circle the length of whose radius is 9 in. Find the length of the tangent drawn from point P to the circle.
14. A tangent and a secant are drawn to a circle from an external point. The length of the tangent is 4 inches. If the length of the internal segment of the secant is 6 inches, find the length of its external segment.
15. A tangent and a secant are drawn to a circle from an external point. The length of the tangent is represented by $3x$. The length of the external segment of the secant is represented by $x + 5$ and the length of its internal segment by $8x - 23$. Find the length of the tangent and the length of the secant.
16. Diameter $\overline{AB}$ of circle O is 21 inches in length. $\overline{AB}$ is extended through B to a point C and tangent $\overline{CD}$ is drawn, meeting the circle at D. $\overline{CD}$ is 6 inches longer than $\overline{BC}$.
    a. Let x represent the length of $\overline{BC}$, and express CD and CA in terms of x.
    b. Express as an equation involving the variable x the relationship that exists among BC, CA, and CD.
    c. Find the length of $\overline{BC}$ by solving the equation obtained in part *b*.
17. *Prove:* If two circles intersect in two points, the lengths of tangents to these circles drawn from any point on the line containing the common chord are equal.
18. $\overline{MN}$ is a diameter of a circle whose center is O. $\overline{OR}$ is a radius perpendicular to $\overline{MN}$. S is a point on radius $\overline{ON}$. $\overline{RS}$ is drawn and extended to meet the circle at W. At W, a tangent is drawn to the circle which meets $\overline{MN}$ extended at T. (*a*) Show that angle RWT and angle WSN are congruent. (*b*) If $TM = 25$ and $TS = 10$, find the length of the radius of the circle.

11. Proportions in the Right Triangle

Projection of a Point or of a Line Segment on a Line

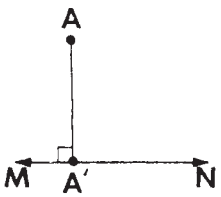


Fig. 6–29

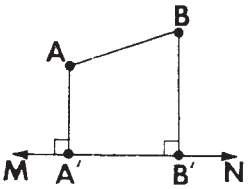


Fig. 6–30

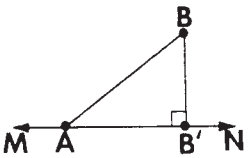


Fig. 6–31

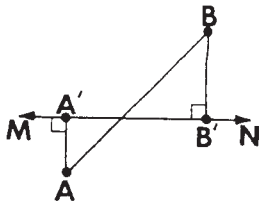


Fig. 6–32

Definition. **The *projection of a given point on a given line* is the foot of the perpendicular drawn from the given point to the given line.**

In Fig. 6–29, point A' is the projection of point A on line $\overleftrightarrow{MN}$.

***Definition.* The *projection of a segment on a given line*, when the segment is not perpendicular to the line, is the segment whose endpoints are the projections of the endpoints of the given line segment on the given line.**

In Fig. 6–30, $\overline{A'B'}$ is the projection of $\overline{AB}$ on $\overleftrightarrow{MN}$.
In Fig. 6–31, $\overline{AB'}$ is the projection of $\overline{AB}$ on $\overleftrightarrow{MN}$.
In Fig. 6–32, $\overline{A'B'}$ is the projection of $\overline{AB}$ on $\overleftrightarrow{MN}$.

NOTE. When the segment is perpendicular to the given line, the projection of the segment on the given line is a point.

Proportions in the Right Triangle

***Theorem 99.* If the altitude is drawn to the hypotenuse of a right triangle,**

***a.* the two triangles thus formed are similar to the given triangle and similar to each other.**

[The proof for this theorem appears on pages 763–764.]

In Fig. 6–33, if ABC is a right triangle and $\overline{CD} \perp$ hypotenuse $\overline{AB}$, then $\triangle ACD \sim \triangle ABC$, $\triangle CBD \sim \triangle ABC$, $\triangle ACD \sim \triangle CBD$.

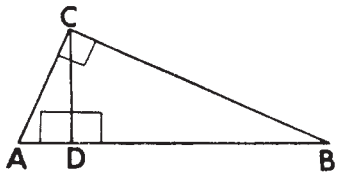


Fig. 6–33

***b.* the length of the altitude is the geometric mean between the lengths of the segments of the hypotenuse.**

In Fig. 6–33, the length of altitude $\overline{CD}$ is the geometric mean between the lengths of $\overline{AD}$ and $\overline{DB}$, the segments of hypotenuse $\overline{AB}$, or $AD:CD = CD:DB$.

***c.* the length of each leg of the given triangle is the geometric mean between the length of the whole hypotenuse and the length of the projection of that leg on the hypotenuse.**

[The proof for this theorem appears on pages 763–764.]

In Fig. 6–33, the length of leg AC is the geometric mean between the length of the hypotenuse $\overline{AB}$ and the length of $\overline{AD}$, the projection of leg $\overline{AC}$ on hypotenuse $\overline{AB}$, or $AB:AC = AC:AD$.

Also, the length of leg $\overline{BC}$ is the geometric mean between the length of hypotenuse $\overline{AB}$ and the length of $\overline{BD}$, the projection of leg $\overline{BC}$ on hypotenuse $\overline{AB}$, or $AB:BC = BC:BD$.

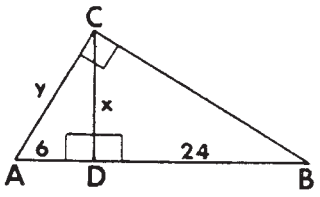
MODEL PROBLEMS

1. In right triangle ABC, altitude $\overline{CD}$ is drawn on hypotenuse $\overline{AB}$. If $AD = 6$ and $DB = 24$, find (*a*) CD and (*b*) AC.

[The solution is given on the next page.]

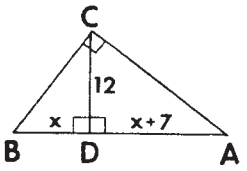
Solution: Since $\overline{CD} \perp$ hypotenuse $\overline{AB}$ in right triangle ABC,

a.	*b.*
1. $\frac{AD}{CD} = \frac{CD}{DB}$	1. $\frac{AD}{AC} = \frac{AC}{AB}$
Let x = length of $\overline{CD}$.	Let y = length of $\overline{AC}$.
2. $\frac{6}{x} = \frac{x}{24}$	2. $\frac{6}{y} = \frac{y}{30}$
3. $x^2 = 144$	3. $y^2 = 180$
4. $x = 12$	4. $y = \sqrt{180} = \sqrt{36} \cdot \sqrt{5} = 6\sqrt{5}$
Answer: $CD = 12$.	*Answer:* $AC = 6\sqrt{5}$.



2. In right triangle ABC, altitude $\overline{CD}$ is drawn to hypotenuse $\overline{AB}$. If $CD = 12$ and AD exceeds DB by 7, find DB and AD.

Solution:



1. Since $\overline{CD} \perp$ hypotenuse $\overline{AB}$ in right triangle ABC,

   $\frac{AD}{CD} = \frac{CD}{DB}$. Let x = the length of $\overline{DB}$.
   Then $x + 7$ = the length of $\overline{AD}$.

2. $\frac{x+7}{12} = \frac{12}{x}$

3. $x(x + 7) = 12 \times 12$

4. $x^2 + 7x = 144$

5. $x^2 + 7x - 144 = 0$

6. $(x - 9)(x + 16) = 0$

7. $x - 9 = 0$	$x + 16 = 0$
8. $x = 9$	$x = -16$

Reject the negative value.

9. $x + 7 = 9 + 7 = 16$.

Answer: $DB = 9$, $AD = 16$.

EXERCISES

1. The altitude upon the hypotenuse of a right triangle divides the hypotenuse into segments of lengths 3 and 12. Find the length of the altitude.
2. The altitude to the hypotenuse of a right triangle divides the hypotenuse into segments of lengths 2 and 5. Find the length of the altitude.

3. In right triangle ABC, altitude $\overline{CD}$ is drawn on hypotenuse $\overline{AB}$. If $CD = 6$ and $AD = 3$, find DB.
4. In right triangle ABC, altitude $\overline{CD}$ is drawn on hypotenuse $\overline{AB}$. If $DB = 5$ and $CD = 10$, find AB.
5. In triangle ABC, angle C is a right angle and $\overline{CD}$ is the altitude on $\overline{AB}$. If $AC = 6$ and $AB = 9$, find AD.
6. In right triangle ABC, the right angle is at C and $\overline{CD}$ is the altitude on $\overline{AB}$. If $AD = 3$ and $DB = 9$, find AC.
7. In a right triangle whose hypotenuse measures 50, the shorter leg measures 30. Find the measure of the projection of the shorter leg on the hypotenuse.
8. In a right triangle whose hypotenuse measures 10, the projection of the longer leg on the hypotenuse measures 6.4. Find the measure of the longer leg.
9. The segments made by the altitude on the hypotenuse of right triangle ABC measure 4 and 5. Find the measure of the shorter leg of triangle ABC.
10. The altitude drawn to the hypotenuse of a right triangle is 8 inches long. If the lengths of the segments of the hypotenuse are represented by x and $4x$, find the number of inches in the smaller segment.
11. If the altitude to the hypotenuse of a right triangle measures 8, the segments of the hypotenuse formed by the altitude may measure (1) 8 and 12 (2) 2 and 32 (3) 3 and 24 (4) 6 and 8.
12. ABC is a right triangle with $\overline{CD}$ the altitude on hypotenuse $\overline{AB}$. If $AC = 20$ and $AB = 25$, find AD, CD, and BC.
13. In right triangle ABC, $\overline{CD}$ is the altitude drawn to hypotenuse $\overline{AB}$. If $CD = 6$, $AD = 3$, and $DB = 5x - 3$, find x.
14. $\overline{CD}$ is the altitude on hypotenuse $\overline{AB}$ of right triangle ABC. $AC = 12$ and $AD = 6$. If BD is represented by x, write an equation that can be used to find x. Solve this equation for x.
15. In right triangle ABC, $\overline{CD}$, the altitude to hypotenuse $\overline{AB}$, measures 3. If DB exceeds AD by 8, find AD and DB.
16. $\overline{BD}$ is the altitude on hypotenuse $\overline{AC}$ of right triangle ABC. $BD = 4$ and $AC = 10$. If CD, the length of the shorter segment of $\overline{AC}$, is represented by x, write an equation which could be used to find x. Solve this equation for x, and find CD and AD.
17. Given right triangle ABC with hypotenuse $\overline{AB}$ and with $\overline{CD}$ the altitude to hypotenuse $\overline{AB}$. $AD = 21$ and $CB = 10$. [Leave irrational answers in radical form.]
    a. Represent DB by x and write an equation that can be used to find x.
    b. Solve for x the equation written in answer to *a*.
    c. Find the length of $\overline{CD}$.

12. The Pythagorean Theorem and Its Applications

Theorem 100. **In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.**

[The proof for this theorem appears on page 765.]

In right triangle ABC (Fig. 6–34), the length of whose hypotenuse is c and the lengths of whose legs are a and b,

$$(\text{hypotenuse})^2 = (\text{one leg})^2 + (\text{the other leg})^2, \text{ or}$$
$$c^2 = a^2 + b^2$$

This relationship is known as the *Theorem of Pythagoras.*

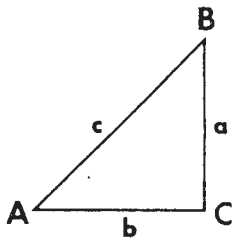


Fig. 6–34

Pythagorean Triples

Let us find the length of the hypotenuse of a right triangle whose legs measure 3 and 4. (See Fig. 6–35.)

1. $c^2 = a^2 + b^2$
2. $c^2 = (3)^2 + (4)^2$
3. $c^2 = 9 + 16$
4. $c^2 = 25$
5. $c = 5$

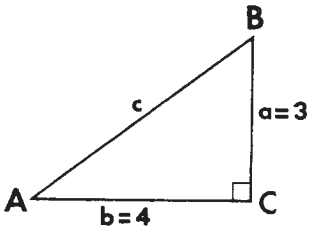


Fig. 6–35

The length of the hypotenuse is 5.

When three integers are so related that the sum of the squares of two of them is equal to the square of the third, the set of three integers is called a *Pythagorean triple.* For example, (3, 4, 5) is called a Pythagorean triple because the numbers 3, 4, and 5 satisfy the relation $a^2 + b^2 = c^2$.

If each number in the triple (3, 4, 5) is doubled, we get the triple (2×3, 2×4, 2×5), or (6, 8, 10). A triangle whose sides are 6, 8, and 10 is similar to a triangle whose sides are 3, 4, and 5 because the corresponding sides of the two triangles are in proportion. Since the triangle whose sides are 3, 4, and 5 is a right triangle, the triangle whose sides are 6, 8, and 10 is also a right triangle. Therefore, (6, 8, 10) is also a Pythagorean triple.

Similarly, (3×3, 3×4, 3×5), or (9, 12, 15), is a Pythagorean triple. Also, (4×3, 4×4, 4×5), or (12, 16, 20), is a Pythagorean triple.

In general, if (3, 4, 5) is a Pythagorean triple, then ($3x$, $4x$, $5x$) is also a Pythagorean triple when x is a positive number.

If we wish to find the hypotenuse of a right triangle whose legs are 30 and 40, we can get the result quickly if we realize that $30 = 10 \times 3$ and $40 = 10 \times 4$. The hypotenuse must be the third number in the Pythagorean triple, 10×5, or 50.

Other examples of Pythagorean triples that occur frequently are:

(5, 12, 13) or, in general, $(5x, 12x, 13x)$, x being a positive integer, and
(8, 15, 17) or, in general, $(8x, 15x, 17x)$, x being a positive integer

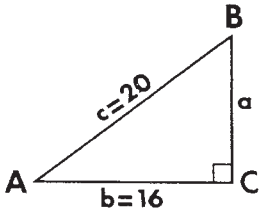
With a knowledge of these Pythagorean triples, we can solve some right triangle problems mentally without writing out a lengthy algebraic solution.

KEEP IN MIND

If the lengths of two sides of a right triangle are known, the length of the third side can always be found by using the Pythagorean Theorem.

MODEL PROBLEMS

1. In a right triangle the length of whose hypotenuse is 20, the length of one leg is 16. Find the length of the other leg.



Solution:

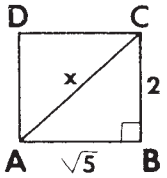
1. $a^2 + b^2 = c^2$
2. $a^2 + (16)^2 = (20)^2$
3. $a^2 + 256 = 400$
4. $a^2 = 144$
5. $a = 12$

Answer: The length of the other leg is 12.

NOTE. If we notice that the hypotenuse, 20, is 4×5 and one leg, 16, is 4×4, then the other leg must be 4×3, or 12, in order to complete the Pythagorean triple $(3x, 4x, 5x)$, x being 4.

2. Find the length of the diagonal of a rectangle whose sides are $\sqrt{5}$ inches and 2 inches respectively.

Solution: Let x = the length of the diagonal.



1. Since $\triangle ABC$ is a right triangle whose hypotenuse is x,

$x^2 = (2)^2 + (\sqrt{5})^2 \qquad (\sqrt{5})^2 = \sqrt{5} \cdot \sqrt{5} = \sqrt{25} = 5.$

[The solution continues on the next page.]

2. $x^2 = 4 + 5$

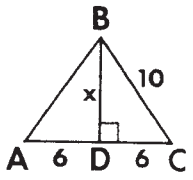
3. $x^2 = 9$

4. $x = 3$

Answer: The length of the diagonal of the rectangle is 3 inches.

3. In an isosceles triangle, the length of each of the congruent sides is 10 and the length of the base is 12. Find the length of the altitude drawn to the base.

Solution: In isosceles triangle ABC, altitude $\overline{BD} \perp \overline{AC}$, and $\overline{BD}$ bisects $\overline{AC}$. Therefore, $\triangle BDC$ is a right triangle and $DC = 6$. Let $x =$ the length of altitude $\overline{BD}$.



1. $(x)^2 + (6)^2 = (10)^2$

2. $x^2 + 36 = 100$

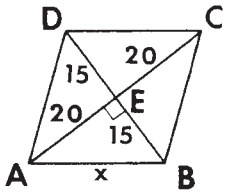
3. $x^2 = 64$

4. $x = 8$

Answer: The length of the altitude drawn to the base is 8.

4. The lengths of the diagonals of a rhombus are 30 and 40. Find the perimeter of the rhombus.

Solution: Since the diagonals of a rhombus are perpendicular to each other and bisect each other, $\triangle AEB$ is a right triangle in which $EB = \frac{1}{2}(30)$, or 15, and $AE = \frac{1}{2}(40)$, or 20.



Let $x =$ the length of hypotenuse $\overline{AB}$.

1. $x^2 = (20)^2 + (15)^2$

2. $x^2 = 400 + 225$

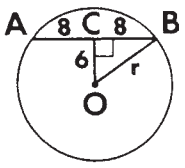
3. $x^2 = 625$

4. $x = 25$

5. Since the four sides of a rhombus are equal in length, the perimeter $= 4 \times 25$, or 100.

Answer: The perimeter of the rhombus is 100.

5. A chord 16 inches long is 6 inches from the center of a circle. Find the length of the radius of the circle.



Solution: Since the distance of a chord from the center of a circle is measured on a line which is perpendicular to the chord and which passes through the center of the circle, $\overleftrightarrow{OC} \perp \overline{AB}$ and $\triangle OCB$ is a right triangle. $\overline{OC}$ bisects $\overline{AB}$. Therefore, $CB = \frac{1}{2}(16) = 8$.

Let r = the length of radius $\overline{OB}$.

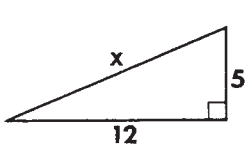
1. $(r)^2 = (6)^2 + (8)^2$
2. $r^2 = 36 + 64$
3. $r^2 = 100$
4. $r = 10$

Answer: The length of the radius of the circle is 10 inches.

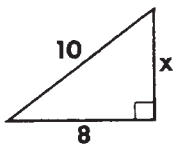
EXERCISES

In the following exercises, all irrational answers may be left in radical form:

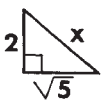
In 1–20, use the information that is marked on the figure to find the value of x.



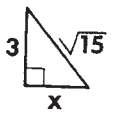
1.



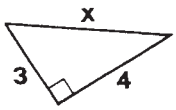
2.



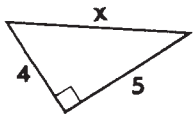
3.



4.

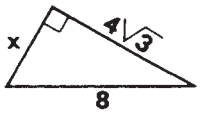


5.

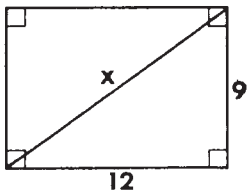


6.

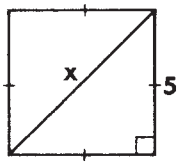
7.



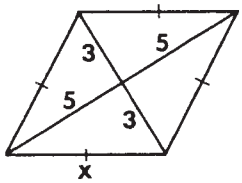
8.



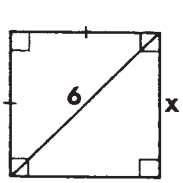
9.



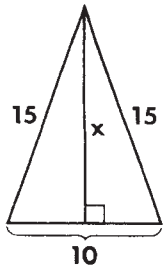
10.



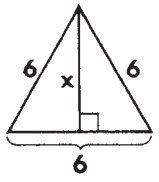
11.



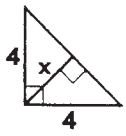
12.



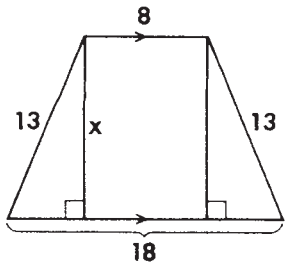
13.



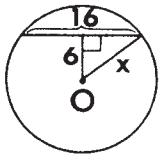
14.



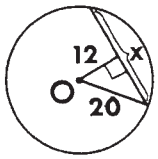
15.



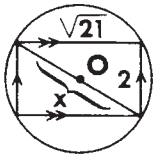
16.



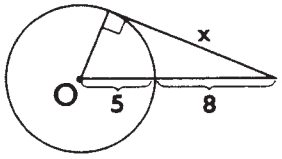
17.



18.



19.



20.

21. Find the length of the hypotenuse of a right triangle whose legs measure 9 and 12.

22. Find the measure of the hypotenuse of a right triangle whose legs measure 5 in. and 3 in.

23. The length of the hypotenuse of a right triangle is 2.5 and the length of one leg is 1.5. Find the length of the other leg.

24. The legs of a right triangle measure 3 in. and 4 in. One leg of another right triangle measures 4 in. and its hypotenuse measures 5 in. (*a*) Are the triangles congruent? (*b*) Why?

25. Find the measure of the diagonal of a rectangle whose sides measure 15 in. and 8 in.

26. If the length of the diagonal of a rectangle is 26 and the length of its base is 24, find the length of the altitude of the rectangle.

27. Find the length of the diagonal of a square whose side is 6 in. in length.

28. Find the length of the side of the rhombus whose diagonals measure 8 and 6.

29. The lengths of the diagonals of a rhombus are 16 and 12. Find the perimeter of the rhombus.

30. Find the length of the side of a square inscribed in a circle whose diameter measures 6.

31. In a circle whose diameter is $\overline{AB}$, chords $\overline{AC}$ and $\overline{BC}$ are drawn. If AB is 6 inches and AC is 3 inches, find in radical form the number of inches in the length of chord $\overline{BC}$.

32. The legs of a right triangle measure 6 inches and 8 inches. Find the number of inches in the length of the radius of the circumscribed circle.

33. Two radii of a circle, $\overline{OA}$ and $\overline{OB}$, are perpendicular to each other and chord $\overline{AB}$ is drawn. If AB is 6, the length of the radius of the circle is
(1) 3 (2) 6 (3) $3\sqrt{2}$ (4) $6\sqrt{2}$

34. In a circle whose radius measures 5 in., a chord is drawn perpendicular to a diameter and at a distance of 3 in. from the center. Find the length of the chord in inches.

35. The radius of a circle measures 13 in. and a chord of this circle measures 10 in. Find the distance of this chord from the center of the circle.

36. A chord 12 in. long is 8 in. from the center of a circle. Find the length of the diameter of the circle.

37. A chord 24 in. long is 5 in. from the center of a circle. Find the length of a chord which is 12 in. from the center of the circle.

38. A point is 15 in. from the center of a circle the length of whose radius is 9 in. Find the length of the tangent drawn from this point to the circle.

39. Find the length of the radius of a circle circumscribed about a rectangle whose base measures 6 and whose altitude measures 8.

40. The lengths of the radii of two concentric circles are 13 in. and 5 in. Find the length of a chord of the larger circle which is tangent to the smaller circle.

41. A point P on the circumference of a circle is joined by line segments to the ends of diameter $\overline{CD}$. If $PC = 12$ and $PD = 16$, find the length of the radius of the circle.

42. The congruent sides of an isosceles triangle are each 15 in. and the base is 24 in. Find the length of the altitude drawn to the base.

43. In triangle ABC, $m\angle C = 90$, $\overline{CM}$ is the median to $\overline{AB}$, and $\overline{MD}$ is perpendicular to $\overline{CB}$. If $CM = 10$ and $MD = 8$, find the perimeter of triangle ABC.

44. In an isosceles trapezoid, the lengths of the bases are 14 and 30 and the length of each of the nonparallel sides is 10. Find the length of the altitude of the trapezoid.

45. In the isosceles trapezoid $ABCD$, angle A contains 45°, the longer base measures 17, and the shorter base measures 7. Find the length of the diagonal $\overline{BD}$.

46. ABC is a right triangle with $\overline{CD}$ the altitude on hypotenuse $\overline{AB}$. If $AC = 20$ and $CD = 12$, find AD.

47. $\overline{AB}$ is a common external tangent to nonintersecting circles O and O', point A being on circle O' and point B being on circle O. $OO' = 17$, $O'A = 10$, and $OB = 2$. Find AB. [*Hint:* Draw $\overline{OE} \perp \overline{O'A}$.]

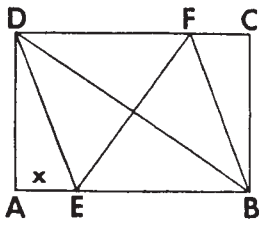
48. In right triangle ABD, the length of leg $\overline{AB}$ is 12 and the length of hypotenuse $\overline{AD}$ is 15. C is a point on $\overline{DB}$. The length of $\overline{AC}$ is 13. Find the length of $\overline{DC}$.

49. In right triangle ABC, the length of leg $\overline{CB}$ is 8 and hypotenuse $\overline{AB}$ is 4 inches longer than leg $\overline{AC}$. (*a*) If the length of $\overline{AC}$ is represented by x, write an equation which can be used to find x. (*b*) Find AC and AB.

50. In right triangle ABC, the length of hypotenuse $\overline{AC}$ is 5 inches. If BC is 1 inch more than AB, find AB and BC.

51. In parallelogram $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. $BC = 26$, $BE = 5x - 5$, $DE = 3x + 1$, $AE = 2y + 2$, and $CE = 4y - 20$. (*a*) Find x and y. (*b*) Show that parallelogram $ABCD$ is a rhombus.

52. $ABCD$ is a rectangle in which $AB = 32$ and $AD = 24$. $\overline{AE} \cong \overline{CF}$, and $\overline{BE} \cong \overline{ED}$. (*a*) Prove $EDFB$ is a rhombus. (*b*) Find DB. (*c*) If AE is represented by x, express ED in terms of x. [*Hint:* Use the relationship $\overline{BE} \cong \overline{ED}$.] (*d*) Find x. (*e*) Find EF.



Ex. 52

13. The Converse of the Pythagorean Theorem

Theorem 101. **If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle.**

In triangle ABC (Fig. 6–36), if a, b, and c represent the lengths of the sides, and if $c^2 = a^2 + b^2$, then triangle ABC is a right triangle, angle C being the right angle.

Theorem 101, which is the converse of the Pythagorean Theorem, can be used to determine whether a triangle is a right triangle.

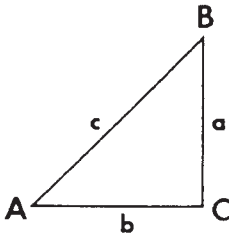


Fig. 6–36

MODEL PROBLEM

The lengths of the sides of a triangle are 8, 15, 17. Show that the triangle is a right triangle.

Solution: Let $c = 17$, the longest side of the triangle, $a = 8$, and $b = 15$.

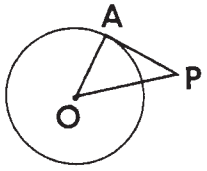
Then $c^2 = (17)^2 = 289$.
Also, $a^2 + b^2 = (8)^2 + (15)^2 = 64 + 225 = 289$.
Since $(17)^2 = (8)^2 + (15)^2$, the triangle is a right triangle.

EXERCISES

In 1–6, show that the triangle the measures of whose sides are given is a right triangle.

1. 15, 20, 25
2. 36, 48, 60
3. 16, 30, 34
4. $\sqrt{5}$, $\sqrt{5}$, $\sqrt{10}$
5. 9, $\sqrt{19}$, 10
6. 4, 8, $4\sqrt{3}$

7. In a parallelogram, the lengths of two adjacent sides are 21 and 28. If the length of a diagonal of the parallelogram is 35, show that the parallelogram is a rectangle.
8. If each side of a rhombus is 3 and a diagonal is $3\sqrt{2}$, show that the rhombus is a square.
9. In circle O, the length of the radius $\overline{OA}$ is 4, $AP = \sqrt{20}$, and $OP = 6$. Show that $\overleftrightarrow{PA}$ is tangent to circle O.



Ex. 9

14. The 30°–60° Right Triangle

In right triangle ABC (Fig. 6–37), $m\angle A = 30$, $m\angle B = 60$, $m\angle C = 90$. In such a triangle, the following relationships can be proved:

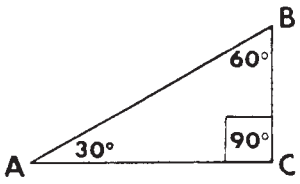


Fig. 6–37

1. The length of $\overline{BC}$, the leg opposite the 30° angle, is equal to one-half the length of the hypotenuse, $\overline{AB}$. $BC = \frac{1}{2}AB$.
2. The length of $\overline{AC}$, the leg opposite the 60° angle, is equal to one-half the length of the hypotenuse $\overline{AB}$, times $\sqrt{3}$. $AC = \frac{1}{2}AB\sqrt{3}$.
3. The length of the longer leg $\overline{AC}$ is equal to the length of the shorter leg $\overline{BC}$, times $\sqrt{3}$. $AC = BC\sqrt{3}$.
4. The length of the shorter leg $\overline{BC}$ is equal to the length of the longer leg $\overline{AC}$, divided by $\sqrt{3}$. $BC = \dfrac{AC}{\sqrt{3}}$.
5. The ratio of the length of the shorter leg $\overline{BC}$ to the length of the hypotenuse $\overline{AB}$ is 1:2. $BC:AB = 1:2$.

Since an altitude of an equilateral triangle divides the triangle into two 30°–60° right triangles (Fig. 6–38), the following relationship in an equilateral triangle can also be proved:

6. The length of the altitude, h, is equal to one-half the length of the side, s, times $\sqrt{3}$. $h = \dfrac{s}{2}\sqrt{3}$.

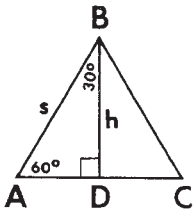
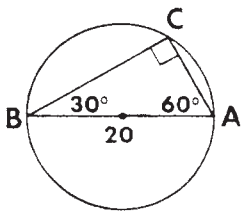


Fig. 6–38

MODEL PROBLEMS

1. In a circle, angle ABC, which is formed by diameter $\overline{AB}$ and chord $\overline{BC}$, contains 30°. If the length of the diameter of the circle is 20, find the length of chord $\overline{AC}$, and find the length of chord $\overline{BC}$ in radical form.



Solution:

1. Since $\overline{AB}$ is a diameter, $m\angle C = 90$.
2. $m\angle B = 30$ and $m\angle A = 60$.
3. $AC = \frac{1}{2}AB = \frac{1}{2}(20) = 10$.
4. $BC = \frac{1}{2}AB\sqrt{3} = \frac{1}{2}(20)\sqrt{3} = 10\sqrt{3}$.

Answer: $AC = 10$, $BC = 10\sqrt{3}$.

2. In triangle ABC, $m\angle A = 30$, $m\angle B = 60$, $m\angle C = 90$, $\overline{AC}$ measures 6 inches. Find BC to the *nearest tenth of an inch.*

Solution:

Method 1

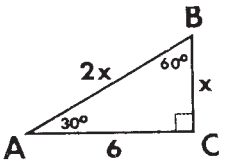
BC, the length of the leg opposite the 30° angle, is equal to one-half of AB, the length of the hypotenuse.

1. Let x = the length of $\overline{BC}$.
2. Then $2x$ = the length of $\overline{AB}$.
3. $(2x)^2 = (x)^2 + (6)^2$
4. $4x^2 = x^2 + 36$
5. $3x^2 = 36$
6. $x^2 = 12$
7. $x = \sqrt{12}$
8. $x = \sqrt{4} \cdot \sqrt{3}$
9. $x = 2\sqrt{3}$

   [Use $\sqrt{3} = 1.73$.]
10. $x = 2(1.73) = 3.46$
11. $x = 3.5$

Method 2

The length of the leg opposite the 30° angle is equal to the length of the leg opposite the 60° angle divided by $\sqrt{3}$.



1. $BC = \dfrac{AC}{\sqrt{3}}$
2. $BC = \dfrac{6}{\sqrt{3}}$
3. $BC = \dfrac{6}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$
4. $BC = \dfrac{6\sqrt{3}}{3}$
5. $BC = 2\sqrt{3}$
6. $BC = 2(1.73) = 3.46$
7. $BC = 3.5$

Answer: $BC = 3.5$ in. to the nearest tenth of an inch.

3. Find in radical form the length of the altitude of an equilateral triangle whose side measures 12 inches.

Solution:

Method 1

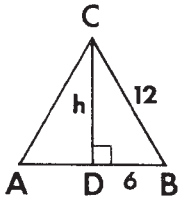
In an equilateral triangle, the length of the altitude, h, is equal to one-half the length of the side, s, times $\sqrt{3}$.

1. $h = \frac{1}{2}s\sqrt{3} \quad s = 12$
2. $h = \frac{1}{2}(12)\sqrt{3}$
3. $h = 6\sqrt{3}$

Method 2

Draw altitude $\overline{CD}$, which bisects base $\overline{AB}$. Let h = the length of the altitude.

In right triangle CDB,



1. $(h)^2 + (6)^2 = (12)^2$
2. $h^2 + 36 = 144$
3. $h^2 = 108$
4. $h = \sqrt{108} = \sqrt{36} \cdot \sqrt{3} = 6\sqrt{3}$

Answer: Length of the altitude $= 6\sqrt{3}$ inches.

EXERCISES

In the following exercises, an irrational answer may be left in radical form unless otherwise indicated:

1. In right triangle ABC, $m\angle A = 30$, $m\angle B = 60$, $m\angle C = 90$. Find BC and AC when AB is:
a. 6 *b.* 10 *c.* 5 *d.* $8\sqrt{3}$ *e.* $4x$

2. In right triangle ABC, $m\angle A = 30$, $m\angle B = 60$, $m\angle C = 90$. Find AB and AC when BC is:
a. 4 *b.* 8 *c.* 3.5 *d.* $4\sqrt{3}$ *e.* $2x$

3. In right triangle ABC, $m\angle A = 30$, $m\angle B = 60$, $m\angle C = 90$. Find BC and AB when AC is:
a. $4\sqrt{3}$ *b.* $7\sqrt{3}$ *c.* $2.5\sqrt{3}$ *d.* $x\sqrt{3}$

4. In triangle ABC, $m\angle B = 30$ and $AB = 6$. Find the length of the altitude $\overline{AD}$ upon side $\overline{BC}$.

5. Two sides of a triangle are 10 and 14 and the angle included between these sides measures 30°. Find the length of the altitude on the side whose length is 14.

6. $\overline{AB}$ intersects $\overleftrightarrow{LM}$ at A. If $m\angle MAB = 60$, find the length of the projection of $\overline{AB}$ on $\overleftrightarrow{LM}$ when AB is:
a. 4 *b.* 8 *c.* 7 *d.* $2\sqrt{3}$ *e.* $2x$

7. $\overline{CD}$ intersects $\overleftrightarrow{AB}$ at C. If $m\angle BCD = 30$, find the length of the projection of $\overline{CD}$ on $\overleftrightarrow{AB}$ when CD is:
   a. 6 *b.* 14 *c.* 9 *d.* 13 *e.* 7*c*
8. In a right triangle, the measure of one acute angle is double the measure of the other acute angle. If the length of the shorter leg of the triangle is 3, find the length of the hypotenuse.
9. The measures of the angles of a triangle are in the ratio 1:2:3. The length of the shortest side of the triangle is 4. Find the length of the longest side.
10. Angle ABC formed by diameter $\overline{AB}$ and chord $\overline{BC}$ of a circle contains 30°. If the length of the diameter of the circle is 10, find the length of chord $\overline{AC}$.
11. The measure of one acute angle of a right triangle is double the measure of the other. The length of the longer leg is $5\sqrt{3}$. Find the length of the hypotenuse.
12. If one angle of a right triangle measures 60 and the length of the hypotenuse is 8, find the length of the side opposite the 60° angle, correct to the *nearest tenth.*
13. In a parallelogram, two adjacent sides whose lengths are 6 in. and 16 in. include an angle of 60°. Find the length of the shorter diagonal.
14. The angle formed by two tangents drawn to a circle from an outside point measures 60. If the length of the radius of the circle is 8 in., find the length of each tangent.
15. In an isosceles triangle ABC, the vertex angle C measures 120 and the length of $\overline{AC}$ is 8. (*a*) Find the length of the altitude on $\overline{AB}$. (*b*) Find AB to the *nearest integer.*
16. In a triangle, two adjacent sides which are 8 in. and 15 in. include an angle of 60°. Find the length of the third side.
17. In a rhombus which contains an angle of 60°, the length of each side is 10 in. Find the length of each diagonal.
18. In right triangle ABC, $m\angle C = 90$ and $m\angle ABC = 30$. D is a point on $\overline{CB}$. If $DB = 50$ and $m\angle ADC = 60$, find AC.
19. *Prove:* In a 30°–60° right triangle, the length of the side opposite the 30° angle is equal to one-half the length of the hypotenuse. [*Hint:* Start with equilateral triangle ABC and draw the bisector of $\angle ACB$.]
20. *Prove:* In a 30°–60° right triangle, the length of the side opposite the 60° angle is equal to one-half the length of the hypotenuse times $\sqrt{3}$. [*Hint:* Start with right triangle ABC in which $m\angle A = 30$ and $m\angle B = 60$. Represent the length of $\overline{AB}$ by h and the length of $\overline{BC}$ by $\frac{1}{2}h$. Then $(x)^2 + (\frac{1}{2}h)^2 = (h)^2$.]
21. Find the length of the altitude of an equilateral triangle the length of whose side is:
   a. 2 *b.* 4 *c.* 8 *d.* 10 *e.* 5 *f.* 9

22. Find the length of the side of an equilateral triangle the length of whose altitude is:

a. $3\sqrt{3}$ *b.* $4\sqrt{3}$ *c.* $6\sqrt{3}$ *d.* $\sqrt{3}$ *e.* $3.5\sqrt{3}$ *f.* $\frac{5}{2}\sqrt{3}$

23. *Prove:* In an equilateral triangle, the length of an altitude is equal to one-half the length of a side times $\sqrt{3}$. [*Hint:* Use one of the 30°–60° right triangle relationships.]

15. The Isosceles Right Triangle

In right triangle ABC (Fig. 6–39), $m\angle A = 45$, $m\angle B = 45$, $m\angle C = 90$. In such a triangle, which is an isosceles right triangle, the following relationships can be proved:

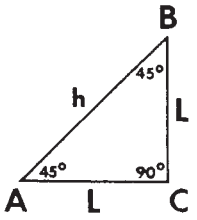


Fig. 6–39

1. The lengths of the legs, $\overline{AC}$ and $\overline{BC}$, are equal. $AC = BC$.
2. The length of the hypotenuse, h, is equal to the length of the leg, L, times $\sqrt{2}$. $h = L\sqrt{2}$.
3. The length of a leg, L, is equal to one-half the length of the hypotenuse, h, times $\sqrt{2}$. $L = \frac{1}{2}h\sqrt{2}$.

Since a diagonal divides a square into two isosceles right triangles (Fig. 6–40), the following relationships can also be proved:

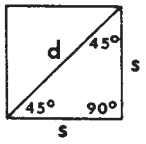


Fig. 6–40

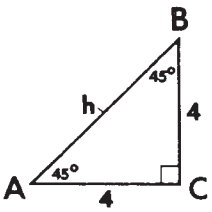
4. The length of the diagonal, d, is equal to the length of the side, s, times $\sqrt{2}$. $d = s\sqrt{2}$.
5. The length of the side, s, is equal to one-half the length of the diagonal, d, times $\sqrt{2}$. $s = \frac{1}{2}d\sqrt{2}$.

MODEL PROBLEMS

1. Find in radical form the length of the hypotenuse of an isosceles right triangle, each of whose legs is 4 units long.

Solution:

Method 1	*Method* 2
In an isosceles right triangle, the length of the hypotenuse is equal to the length of a leg times $\sqrt{2}$.	In right $\triangle ABC$,
1. $h = L\sqrt{2}$ $L = 4$.	1. $h^2 = (4)^2 + (4)^2$
2. $h = 4\sqrt{2}$	2. $h^2 = 16 + 16$
	3. $h^2 = 32$
	4. $h = \sqrt{32} = \sqrt{16} \cdot \sqrt{2}$ $= 4\sqrt{2}$

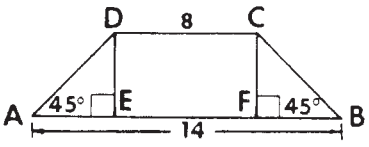


Answer: The length of the hypotenuse is $4\sqrt{2}$.

2. The lengths of the bases of an isosceles trapezoid are 8 and 14, and each of the base angles measures 45°. Find the length of the altitude of the trapezoid.

Solution:

1. Draw altitudes $\overline{DE}$ and $\overline{CF}$.
2. $DCFE$ is a rectangle.
3. $DC = EF = 8$.



4. Since $\triangle AED \cong \triangle BFC$, $AE = BF$. Then $AE = \frac{1}{2}(AB - EF)$
$= \frac{1}{2}(14 - 8)$
$= \frac{1}{2}(6)$
$= 3$
5. Since $\triangle AED$ is a 45°–45°–90° triangle, $AE = DE = 3$.

Answer: The length of the altitude is 3.

3. Find to the nearest tenth of an inch the length of a side of a square whose diagonal measures 8 inches.

Solution:

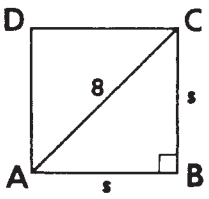
Method 1

In a square, the length of a side is equal to one-half the length of a diagonal times $\sqrt{2}$.

1. $s = \frac{1}{2}d\sqrt{2}$ $d = 8$.
2. $s = \frac{1}{2}(8)\sqrt{2}$
3. $s = 4\sqrt{2}$
[Use $\sqrt{2} = 1.41$.]
4. $s = 4(1.41) = 5.64$
5. $s = 5.6$

Method 2

Let s = the length of a side of the square.
In right $\triangle ABC$,



1. $s^2 + s^2 = 8^2$
2. $2s^2 = 64$
3. $s^2 = 32$
4. $s = \sqrt{32} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$
[Use $\sqrt{2} = 1.41$.]
5. $s = 4(1.41) = 5.64$
6. $s = 5.6$

Answer: The length of a side of the square is 5.6 inches, to the nearest tenth of an inch.

EXERCISES

In the following exercises, an irrational answer may be left in radical form unless otherwise indicated:

1. In right triangle ABC, $m\angle A = 45$ and $m\angle B = 45$. Find AC and AB when BC is:
   a. 6 *b.* 8 *c.* 12 *d.* 5 *e.* $4x$
2. In right triangle ABC, $m\angle A = 45$ and $m\angle B = 45$. Find BC and AC when AB is:
   a. 6 *b.* 8 *c.* 5 *d.* $4\sqrt{2}$ *e.* $2x$
3. The length of a leg of an isosceles right triangle is 4. Find the length of the hypotenuse to the *nearest tenth.*
4. In an isosceles right triangle, the length of one leg is 10. Express in radical form the length of the altitude upon the hypotenuse.
5. $\overline{AB}$ intersects $\overleftrightarrow{LM}$ at A. If $m\angle MAB = 45$, find the length of the projection of $\overline{AB}$ on $\overleftrightarrow{LM}$ when AB is:
   a. 4 *b.* 12 *c.* 3 *d.* $6\sqrt{2}$ *e.* $4x$
6. The lengths of the bases of an isosceles trapezoid are 7 and 15. Each leg makes an angle of 45° with the longer base. Find the length of the altitude of the trapezoid.
7. The bases of an isosceles trapezoid are 9 and 15, and each base angle contains 45°. Find the length of the altitude of the trapezoid.
8. *Prove:* In an isosceles right triangle, the length of the hypotenuse is equal to the length of a leg times $\sqrt{2}$.
9. *Prove:* In an isosceles right triangle, the length of a leg is equal to one-half the length of the hypotenuse times $\sqrt{2}$.
10. Find the length of the diagonal of a square whose side is:
    a. 2 *b.* 8 *c.* 5 *d.* 9 *e.* $4\sqrt{2}$ *f.* $3\sqrt{2}$ *g.* $6x$
11. Find the length of the side of a square whose diagonal is:
    a. 4 *b.* 6 *c.* 3 *d.* 5 *e.* $6\sqrt{2}$ *f.* $2x$ *g.* $4a\sqrt{2}$
12. Find to the *nearest tenth of an inch* the length of the diagonal of a square whose side is 3 inches in length.
13. Find to the *nearest tenth of an inch* the length of the side of a square whose diagonal is 10 inches in length.
14. *Prove:* The length of the diagonal of a square is equal to the length of the side of the square times $\sqrt{2}$.
15. *Prove:* The length of the side of a square is equal to one-half the length of the diagonal of the square times $\sqrt{2}$.

16. Trigonometry of the Right Triangle

The word *trigonometry*, which comes from the Greek, means "measurement of triangles." In this branch of mathematics, we will study relationships among the measures of the sides and angles of triangles.

Direct and Indirect Measurement

Many mathematical problems involve the measurement of line segments and angles. Sometimes we can conveniently make a *direct measurement* of a segment or an angle by applying the unit of measure to it. The number of times the unit of measure is contained in the segment or angle represents the measure of the segment or angle.

However, in many situations, it is inconvenient or impossible to apply the unit of measure directly to the object being measured; for example, we cannot measure directly the height of a tall tree or building, the width of a river, or the distance to the sun. In such cases, we resort to methods of *indirect measurement.*

When we use indirect measurement to discover the length of a line segment, for example, we first measure directly segments and angles which can be conveniently measured. Then we compute the length of the segment we wish to measure by using a formula or mathematical relationship which relates the length of that segment with the measurements which were made directly. When we worked with similar figures and the Pythagorean Theorem, we used indirect measurement.

In our study of trigonometry of the right triangle, we will discover new relationships which will provide additional methods for measuring segments and angles indirectly. Engineers, surveyors, physicists, and astronomers frequently use these trigonometric methods in their work.

17. The Tangent Ratio

Each of the triangles in Fig. 6–41 represents a right triangle in which there is a 31° angle. In each figure, the lengths of the leg opposite the 31° angle and the leg adjacent to the 31° angle are shown. In each triangle, let us find the ratio of the length of the leg opposite the 31° angle to the length of the leg adjacent to the 31° angle. Note that in each triangle the indicated linear measures are given correct to the nearest integer.

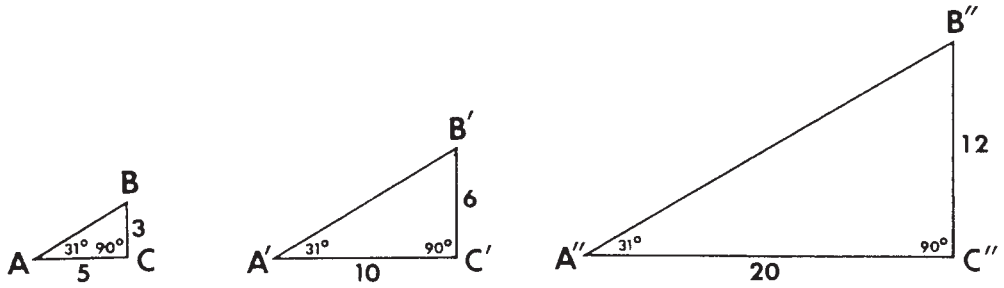


Fig. 6–41

$$\frac{BC}{AC} = \frac{3}{5} = 0.60 \qquad \frac{B'C'}{A'C'} = \frac{6}{10} = 0.60 \qquad \frac{B''C''}{A''C''} = \frac{12}{20} = 0.60$$

Notice that in all three cases, the ratio,

$$\frac{\text{length of the leg opposite the } 31° \text{ angle}}{\text{length of the leg adjacent to the } 31° \text{ angle}} = 0.60, \text{ a constant}$$

We might have expected that the three ratios $\frac{BC}{AC}$, $\frac{B'C'}{A'C'}$, and $\frac{B''C''}{A''C''}$ would be equal because the right triangles ABC, $A'B'C'$, and $A''B''C''$, each of which contains an angle of 31°, are similar, and the ratios of the lengths of corresponding sides in these similar triangles must be equal.

In fact, in all right triangles which contain an angle of 31°, the value of the ratio $\frac{\text{length of the leg opposite the } 31° \text{ angle}}{\text{length of the leg adjacent to the } 31° \text{ angle}}$ is constant (approximately 0.60), no matter what the size of the triangle.

What we have shown to be true for a 31° angle would also be true for any other acute angle in a right triangle. In general, in every right triangle having an acute angle whose measure is a particular number of degrees, the ratio of the length of the leg opposite the acute angle to the length of the leg adjacent to the acute angle is constant. This is true because every right triangle containing an acute angle whose measure is a given number of degrees is similar to every other right triangle containing an acute angle whose measure is the same number of degrees. For different acute angles, the ratio is a different constant.

In Fig. 6–42, in which $\angle C$, $\angle C'$, and $\angle C''$ are right angles, we can see that $\triangle ABC \sim \triangle AB'C' \sim \triangle AB''C''$. Therefore, by using the definition of similar polygons, it follows that:

$$\frac{BC}{AC} = \frac{B'C'}{AC'} = \frac{B''C''}{AC''} = \text{a constant ratio for } \angle A.$$

This ratio is called the *tangent of the angle.*

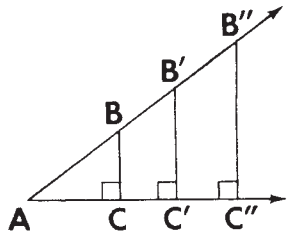


Fig. 6–42

Definition. **The *tangent of an acute angle of a right triangle* is the ratio of the length of the leg opposite the acute angle to the length of the leg adjacent to the acute angle.**

In right triangle ABC (Fig. 6–43), with $m\angle C = 90$, the definition of the *tangent of angle A*, abbreviated "tan A," is:

$$\tan A = \frac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent to } \angle A} = \frac{BC}{AC} = \frac{a}{b}$$

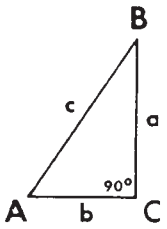


Fig. 6–43

Finding Tangents

As the measure of angle A changes, the tangent ratio for angle A also changes. The tangent ratio for angle A depends upon the measure of angle A, not upon the size of the right triangle which contains angle A. Mathematicians have constructed a table of tangent ratios for all acute angles whose measures are between 0 and 90. This table, which is called a table of trigonometric functions, is found on page 771 in the fourth column.

To find tan 28° from this table, for example, first look in the column headed "Angle" for the angle 28°. Then, in the column headed "Tangent," on the same horizontal line as 28°, find the number 0.5317. Thus, tan 28° = 0.5317 to the *nearest ten-thousandth.*

The table may also be used to find an angle when its tangent ratio is known. Thus, if $\tan A = 1.5399$, we see from the table that angle A must contain 57°.

A calculator can be used to find the tangent ratio of an angle, and the measure of an angle when its tangent ratio is known. By using the calculator we can find tan 60°. Try each of the following methods to find which one will work for your calculator. Be sure that the calculator is in degree mode before you start.

Method 1: Enter: 60 [TAN]

Method 2: Enter: [TAN] 60 [=]

Display: [1.7320508]

Therefore, tan 60° ≈ 1.7321.

If the tangent ratio is known we can use a calculator to find the measure of the angle. For example, let $\tan A = 0.9004$. Try the following methods to find the measure of angle A. Make sure your calculator is in degree mode before you start.

Method 1: Enter: 0.9004 [2nd] [TAN^{-1}]

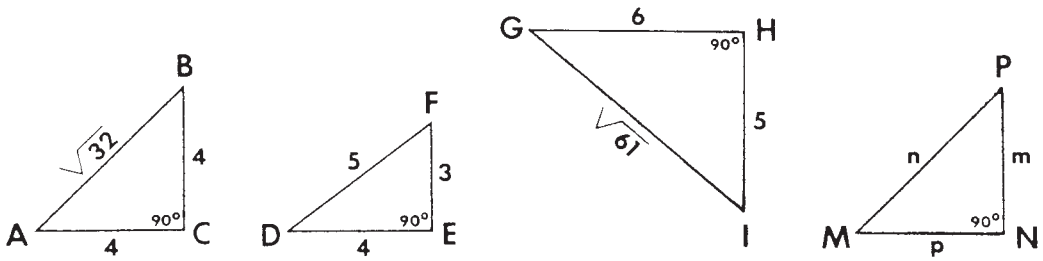
Method 2: Enter: [2nd] [TAN^{-1}] 0.9004 [=]

Display: [41.999872]

The measure of angle A to the *nearest degree* is 42°.

EXERCISES

In 1–4, represent the tangent of each acute angle.



1. 2. 3. 4.

5. In triangle ABC, $m\angle C = 90$, $AC = 6$, and $AB = 10$. Find $\tan A$.
6. In triangle RST, $m\angle T = 90$, $RS = 13$, and $ST = 12$. Find $\tan S$.

Find each of the following:

7. $\tan 10°$	**8.** $\tan 30°$	**9.** $\tan 70°$	**10.** $\tan 45°$
11. $\tan 1°$	**12.** $\tan 89°$	**13.** $\tan 36°$	**14.** $\tan 60°$

Find the measure of angle A if:

15. $\tan A = 0.0875$	**16.** $\tan A = 0.3640$	**17.** $\tan A = 0.5543$
18. $\tan A = 1.0000$	**19.** $\tan A = 2.0503$	**20.** $\tan A = 3.0777$

Find the measure of angle A to the *nearest degree* if:

21. $\tan A = 0.3754$	**22.** $\tan A = 0.7654$	**23.** $\tan A = 1.8000$
24. $\tan A = 0.3500$	**25.** $\tan A = 0.1450$	**26.** $\tan A = 2.9850$

27. Does the tangent of an angle increase or decrease as the measure of the angle varies from 1 to 89?

28. *a.* Find whether or not $\tan 40°$ is twice $\tan 20°$.

    b. If the measure of an angle is doubled, is the tangent of the angle also doubled?

29. In triangle ABC, $m\angle C = 90$, $AC = 6$, and $BC = 6$. (*a*) Find $\tan A$. (*b*) Find the measure of angle A.

30. In triangle RST, $m\angle S = 90$, $TS = 4$, and $RS = 3$. (*a*) Find $\tan T$ to the *nearest ten-thousandth*. (*b*) Find the measure of angle T to the *nearest degree*.

31. In triangle ABC, $m\angle C = 90$, $AC = 5$, and $BC = 12$. (*a*) Find $\tan B$ to the *nearest ten-thousandth*. (*b*) Find the degree measure of angle B to the *nearest degree*.

Angle of Elevation and Angle of Depression

In Fig. 6–44, if a person using a telescope or some similar instrument wishes to sight the top of the telephone pole above him, he must elevate (tilt upward) the instrument from a horizontal position. The line $\overleftrightarrow{OT}$ passing through the eye of the observer, O, and the top of the pole, T, is called the *line of sight*. The angle determined by the rays which are part of the horizontal line and the line of sight, $\angle AOT$, is called the *angle of elevation* of the top of the pole, T, from point O. (The horizontal line and the line of sight must be in the same vertical plane.)

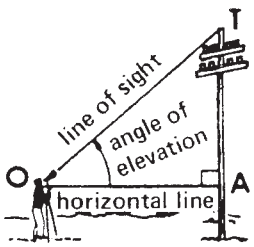


Fig. 6–44

In Fig. 6–45, if a person using a telescope or some similar instrument wishes to sight the boat below him, he must depress (tilt downward) the instrument from a horizontal position. The line $\overleftrightarrow{OB}$ passing through the eye of the observer, O, and the boat, B, is called the *line of sight*. The angle determined by the rays which are part of the horizontal line and the line of sight, $\angle HOB$, is called the *angle of depression* of the boat, B, from point O. (The horizontal line and the line of sight must be in the same vertical plane.)

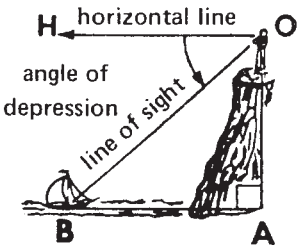


Fig. 6–45

In Fig. 6–45, if we find the measure of the angle of elevation of O from B, $\angle OBA$, and also find the measure of the angle of depression of B from O, $\angle HOB$, we discover that both angles contain the same number of degrees. We therefore say that the angle of elevation of O from B is congruent to the angle of depression of B from O. Note that this must be so because $\overleftrightarrow{HO}$ is parallel to $\overleftrightarrow{AB}$, making the alternate interior angles, $\angle HOB$ and $\angle ABO$, congruent.

Using the Tangent Ratio to Solve Problems

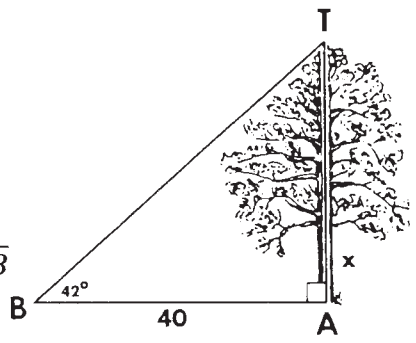
To solve problems by use of the tangent ratio, proceed as follows:

1. Make an approximate scale drawing which contains the line segments and angles given in the problem and those to be found.
2. Select a right triangle in which either (*a*) the measures of two legs are given (known) and the measure of an acute angle is to be found or (*b*) the measures of one leg and an acute angle are given (known) and the measure of the other leg is to be found.
3. Write the formula for the tangent of the acute angle mentioned in step 2, and then substitute in the formula the values given in the problem.
4. Solve the resulting equation.

MODEL PROBLEMS

1. At a point on the ground 40 feet from the foot of a tree, the angle of elevation of the top of the tree contains 42°. Find the height of the tree to the *nearest foot.*

Solution: Since the segments mentioned in the problem are legs of a right triangle opposite and adjacent to the given acute angle, use the tangent ratio.

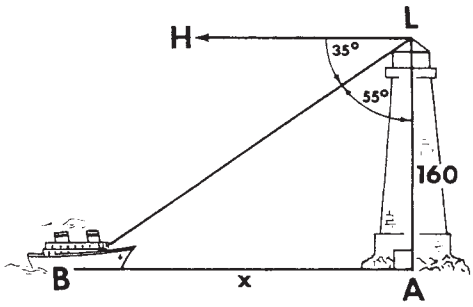


1. $\tan B = \dfrac{\text{length of leg opposite } \angle B}{\text{length of leg adjacent to } \angle B}$
2. $\tan B = \dfrac{AT}{BA}$ Let $x = AT$.
3. $\tan 42° = \dfrac{x}{40}$ Tan 42° = 0.9004.
4. $0.9004 = \dfrac{x}{40}$
5. $x = 40(0.9004)$ [Multiply both members of the equation by 40.]
6. $x = 36.016$

Answer: The height of the tree is 36 feet to the nearest foot.

2. From the top of a lighthouse 160 feet above sea level, the angle of depression of a boat at sea contains 35°. Find to the *nearest foot* the distance from the boat to the foot of the lighthouse.

Solution: The angle of depression, $\angle HLB$, is not inside right triangle BAL. To find the measure of $\angle BLA$, which is inside triangle BAL, subtract $m\angle HLB$ from 90. $90 - 35 = 55$. Since the segments mentioned in the problem are legs of right triangle BAL opposite and adjacent to $\angle BLA$, use the tangent ratio.



[The solution continues on the next page.]

1. $\tan \angle BLA = \dfrac{\text{length of leg opposite } \angle BLA}{\text{length of leg adjacent to } \angle BLA}$

2. $\tan \angle BLA = \dfrac{BA}{LA}$ Let $x = BA$

3. $\tan 55° = \dfrac{x}{160}$ Tan $55° = 1.4281$.

4. $1.4281 = \dfrac{x}{160}$

5. $x = 160(1.4281)$

6. $x = 228.496$

Answer: 228 feet.

3. A ladder which is leaning against a building makes an angle of 75° with the ground. If the top of the ladder reaches a point which is 20 feet above the ground, find to the *nearest foot* the distance from the foot of the ladder to the foot of the building.

Solution:

Method 1

1. $\tan A = \dfrac{BC}{AC}$

   Let $x = AC$.

2. $\tan 75° = \dfrac{20}{x}$

3. $3.7321 = \dfrac{20}{x}$

4. $3.7321x = 20$

5. $x = \dfrac{20}{3.7321}$

6. $x = 5.3$

Answer: 5 feet.

Method 2

1. Find $m\angle B$,

   $m\angle B = 90 - 75 = 15$.

2. $\tan B = \dfrac{AC}{BC}$

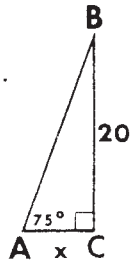
   Let $x = AC$.

3. $\tan 15° = \dfrac{x}{20}$

4. $0.2679 = \dfrac{x}{20}$

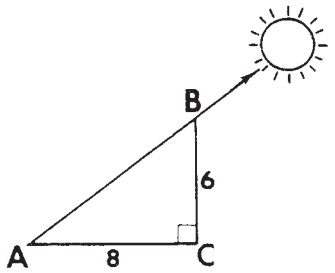
5. $x = 20(0.2679)$

6. $x = 5.3580$



NOTE. In method 1, since the unknown was the measure of the leg adjacent to $\angle A$, the solution required the inconvenient long division $\dfrac{20}{3.7321}$. In method 2, however, when we used the other acute angle, $\angle B$, the unknown was the measure of the leg opposite $\angle B$; and the solution required the convenient multiplication 20(0.2679).

4. Find to the *nearest degree* the measure of the angle of elevation of the sun when a vertical pole 6 feet high casts a shadow 8 feet long.



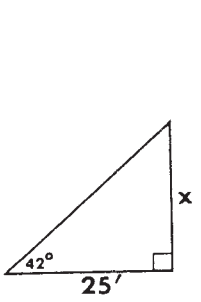
Solution: The angle of elevation of the sun is the same as $\angle A$, the angle of elevation of the top of the pole from A. Since the segments mentioned in the problem are legs of a right triangle opposite and adjacent to $\angle A$, use the tangent ratio.

1. $\tan A = \dfrac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent to } \angle A}$
2. $\tan A = \dfrac{BC}{AC} = \dfrac{6}{8}$ Express $\dfrac{6}{8}$ as the decimal 0.7500.
3. $\tan A = 0.7500$
4. $m\angle A = 37$

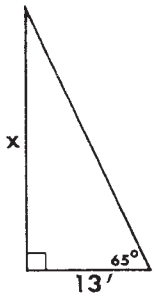
Answer: 37° to the nearest degree.

EXERCISES

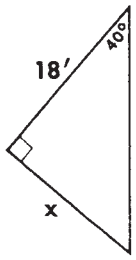
In 1–8, in the given triangle, find the length of the side marked x to the *nearest foot* or the number of degrees contained in the angle marked x to the *nearest degree*.



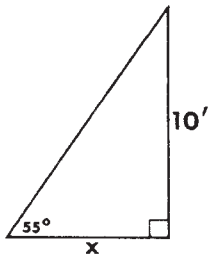
1.



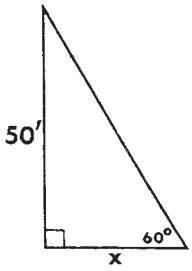
2.



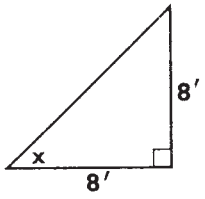
3.



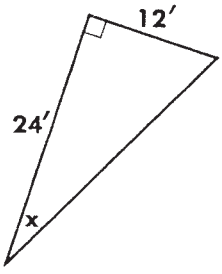
4.



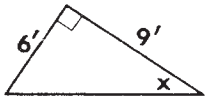
5.



6.



7.



8.

9. At a point on the ground 50 feet from the foot of a tree, the angle of elevation of the top of the tree contains 48°. Find the height of the tree to the *nearest foot.*
10. A ladder is leaning against a wall. The foot of the ladder is 6.5 feet from the wall. The ladder makes an angle of 74° with the level ground. How high on the wall does the ladder reach? Round off the answer to the *nearest tenth of a foot.*
11. A boy visiting New York City views the Empire State Building from a point on the ground, A, which is 940 feet from the foot, C, of the building. The angle of elevation of the top, B, of the building as seen by the boy contains 53°. Find the height of the building to the *nearest foot.*
12. Find to the *nearest foot* the height of a vertical post if its shadow is 18 feet long when the angle of elevation of the sun contains 38°.
13. From the top of a lighthouse 160 feet high, the angle of depression of a boat out at sea is an angle of 24°. Find to the *nearest foot* the distance from the boat to the foot of the lighthouse, the foot of the lighthouse being at sea level.
14. From the top of a tower 80 feet high, the angle of depression of an object on the ground contains 38°. Find to the *nearest foot* the distance from the object to the foot of the tower.
15. Find to the *nearest degree* the measure of the angle of elevation of the sun when a boy 5 feet high casts a shadow 5 feet long.
16. Find to the *nearest degree* the measure of the angle of elevation of the sun when a vertical post 15 feet high casts a shadow 20 feet long.
17. A ladder leans against a building. The top of the ladder reaches a point on the building which is 18 feet above the ground. The foot of the ladder is 7 feet from the building. Find to the *nearest degree* the measure of the angle which the ladder makes with the level ground.

18. The Sine Ratio

Since the tangent ratio involves the two legs of a right triangle, it is not directly useful in solving problems in which the hypotenuse, a leg, and an acute angle are involved. In a case where two of these three parts of a right triangle are given and the third part is to be found, ratios other than the tangent ratio can be more useful.

If we refer to Fig. 6–42 on page 333, we can see that since $\triangle ABC \sim \triangle AB'C' \sim \triangle AB''C''$, it also follows that:

$$\frac{BC}{AB} = \frac{B'C'}{AB'} = \frac{B''C''}{AB''} = \text{a constant ratio for } \angle A.$$

This ratio is called the *sine of the angle.*

Definition. **The *sine of an acute angle of a right triangle* is the ratio of the length of the leg opposite the acute angle to the length of the hypotenuse.**

In right triangle ABC (Fig. 6–46), with $m\angle C = 90$, the definition of the *sine of angle A*, abbreviated "sin A," is:

$$\sin A = \frac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}} = \frac{BC}{AB} = \frac{a}{c}$$

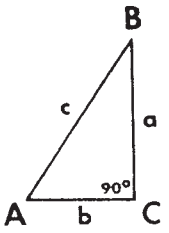


Fig. 6–46

The Table of Sines

As the measure of angle A changes, the sine ratio for angle A also changes. The sine ratio for angle A depends upon the measure of angle A, not upon the size of the right triangle which contains angle A. The table of sine ratios is found on page 771 in the second column.

A calculator can be used to find the sine ratio of an angle, and the measure of an angle when its sine ratio is known. Make sure the calculator is in degree mode. For example, to find sin 50° and the measure of $\angle A$ when $\sin A = 0.2588$, try the following methods.

Method 1: Enter: 50 [SIN]

Method 2: Enter: [SIN] 50 [=]

Display: [0.76604444]

Therefore, $\sin 50° \approx 0.7660$

Enter: 0.2588 [2nd] [SIN^{-1}]

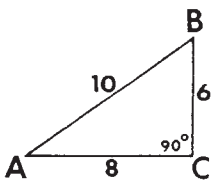
Enter: [2nd] [SIN^{-1}] 0.2588 [=]

Display: [14.9988703]

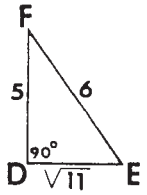
$A \approx 15°$

EXERCISES

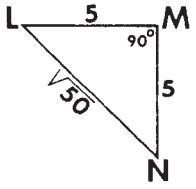
In 1–4, represent the sine of each acute angle.



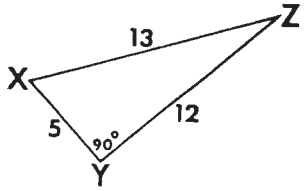
1.



2.



3.



4.

5. In triangle ABC, $m\angle C = 90$, $AC = 4$, and $BC = 3$. Find $\sin A$.
6. In triangle RST, $m\angle S = 90$, $RS = 5$, and $ST = 12$. Find $\sin T$.

Using the table on page 771, find each of the following:

7. sin 18°	**8.** sin 42°	**9.** sin 58°	**10.** sin 76°
11. sin 1°	**12.** sin 89°	**13.** sin 35°	**14.** sin 68°

Find the measure of angle A if:

15. $\sin A = 0.1908$ **16.** $\sin A = 0.8387$ **17.** $\sin A = 0.6561$
18. $\sin A = 0.3420$ **19.** $\sin A = 0.7071$ **20.** $\sin A = 0.9962$

Find the degree measure of angle A to the *nearest degree* if:

21. $\sin A = 0.1900$ **22.** $\sin A = 0.8740$ **23.** $\sin A = 0.5800$
24. $\sin A = 0.9725$ **25.** $\sin A = 0.1275$ **26.** $\sin A = 0.8695$

27. Does the sine of an angle increase or decrease as the measure of the angle varies from 1 through 89?

28. *a.* Find whether or not sin 50° is twice sin 25°.
b. If the measure of an angle is doubled, is the sine of the angle also doubled?

29. In triangle ABC, $m\angle C = 90$, $BC = 20$, and $BA = 40$. (*a*) Find $\sin A$ to the *nearest ten-thousandth.* (*b*) Find the measure of angle A.

30. In triangle ABC, $m\angle C = 90$, $AC = 5$, and $BC = 12$. (*a*) Find $\sin B$ to the *nearest ten-thousandth.* (*b*) Find the degree measure of angle B to the *nearest degree.*

31. Why must the sine of an acute angle be less than 1?

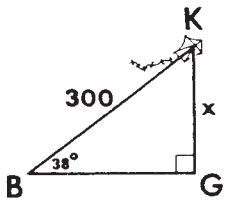
Using the Sine Ratio to Solve Problems

When the hypotenuse, a leg, and an acute angle of a right triangle are involved in a problem, with the measures of any two of these three parts given (known), the use of the sine ratio will help us to find the measure of the unknown third part. We proceed as we did when we were using the tangent ratio.

MODEL PROBLEMS

1. A boy who is flying a kite lets out 300 feet of string which makes an angle of 38° with the ground. Assuming that the string is straight, how high above the ground is the kite? Give your answer correct to the *nearest foot.*

Solution: Since the segments mentioned in the problem are the leg opposite the acute angle and the hypotenuse of a right triangle, use the sine ratio.



1. $\sin B = \dfrac{\text{length of leg opposite } \angle B}{\text{length of hypotenuse}}$

2. $\sin B = \dfrac{KG}{KB}$ Let $x = KG$.

3. $\sin 38° = \dfrac{x}{300}$ Sin 38° = 0.6157.

4. $0.6157 = \dfrac{x}{300}$

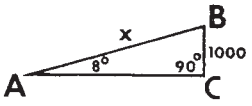
5. $x = 300(0.6157)$

6. $x = 184.71$

Answer: 185 feet to the nearest foot.

2. A road is inclined 8° to the horizontal. Find to the *nearest hundred feet* the distance one must drive up this road to increase one's altitude 1,000 feet.

Solution: Since the segments mentioned in the problem are the leg opposite the acute angle and the hypotenuse of a right triangle, use the sine ratio.



1. $\sin A = \dfrac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}}$

2. $\sin A = \dfrac{BC}{AB}$ Let $x = AB$.

3. $\sin 8° = \dfrac{1{,}000}{x}$ Sin 8° = 0.1392.

4. $0.1392 = \dfrac{1{,}000}{x}$

5. $0.1392x = 1{,}000$

6. $x = \dfrac{1{,}000}{0.1392}$

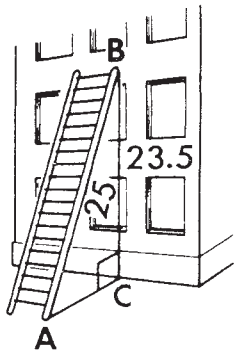
7. $x = 7{,}184$

Answer: 7,200 feet correct to the nearest hundred feet.

3. A ladder 25 feet long leans against a building and reaches a point 23.5 feet above the ground. Find to the *nearest degree* the angle which the ladder makes with the ground.

[The solution is given on the next page.]

Solution: Since the given segments are the hypotenuse of a right triangle and the leg opposite the acute angle to be found, use the sine ratio.

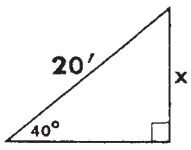


1. $\sin A = \dfrac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}}$
2. $\sin A = \dfrac{23.5}{25}$ Express $\dfrac{23.5}{25}$ as the decimal 0.9400.
3. $\sin A = 0.9400$
4. $m\angle A = 70$

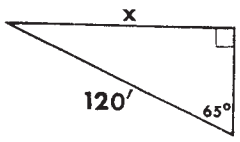
Answer: 70° to the nearest degree.

EXERCISES

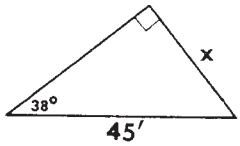
In 1–8, in the given right triangles, find the length of the side marked x to the *nearest foot* or the number of degrees contained in the angle marked x to the *nearest degree.*



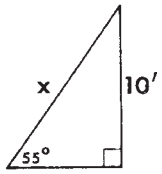
1.



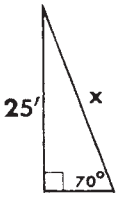
2.



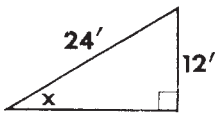
3.



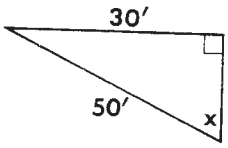
4.



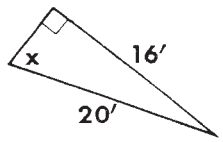
5.



6.



7.



8.

9. A wooden beam 24 feet long leans against a wall and makes an angle of 71° with the ground. Find to the *nearest foot* how high up the wall the beam reaches.
10. A boy who is flying a kite lets out 300 feet of string which makes an angle of 52° with the ground. Assuming that the string is stretched taut, find to the *nearest foot* how high the kite is above the ground.

11. A straight road to the top of a hill is 2500 feet long and makes an angle of 12° with the horizontal. Find the height of the hill to the *nearest hundred feet.*
12. A ladder which leans against a building makes an angle of 75° with the ground and reaches a point on the building 20 feet above the ground. Find to the *nearest foot* the length of the ladder.
13. From an airplane which is flying at an altitude of 3,000 feet, the angle of depression of an airport ground signal is an angle of 27°. Find to the *nearest hundred feet* the distance between the airplane and the airport signal.
14. An airplane climbs at an angle of 11° with the ground. Find to the *nearest hundred feet* the distance it has traveled when it has attained an altitude of 400 feet.
15. A 20-foot pole which is leaning against a wall reaches a point 18 feet above the ground. Find to the *nearest degree* the number of degrees contained in the angle which the pole makes with the ground.
16. In order to reach the top of a hill which is 250 feet high, one must travel 2,000 feet up a straight road which leads to the top. Find to the *nearest degree* the number of degrees contained in the angle which the road makes with the horizontal.
17. After takeoff, a plane flies in a straight line for a distance of 4,000 feet in order to gain an altitude of 800 feet. Find to the *nearest degree* the number of degrees contained in the angle which the rising plane makes with the ground.

19. The Cosine Ratio

A third important ratio in a right triangle involves the length of the leg adjacent to one of the acute angles of the triangle and the length of the hypotenuse.

If we refer again to Fig. 6–42 on page 333, we can see that since $\triangle ABC \sim \triangle AB'C' \sim \triangle AB''C''$, it also follows that:

$$\frac{AC}{AB} = \frac{AC'}{AB'} = \frac{AC''}{AB''} = \text{a constant ratio for } \angle A.$$

This ratio is called the *cosine of the angle.*

Definition. **The *cosine of an acute angle of a right triangle* is the ratio of the length of the leg adjacent to the acute angle to the length of the hypotenuse.**

In right triangle ABC (Fig. 6–47), with $m\angle C = 90$, the definition of the *cosine of angle A*, abbreviated "cos A," is:

$$\cos A = \frac{\text{length of leg adjacent to } \angle A}{\text{length of hypotenuse}} = \frac{AC}{AB} = \frac{b}{c}$$

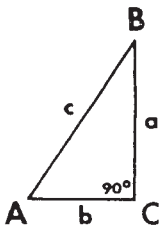


Fig. 6–47

The Table of Cosines

As angle A changes, the cosine ratio for angle A also changes. The cosine ratio for angle A depends upon the measure of angle A, not upon the size of the right triangle which contains angle A. The table of cosine ratios is found on page 771 in the third column.

A calculator can be used to find the cosine ratio of an angle, and the measure of an angle when its cosine ratio is known. Make sure the calculator is in degree mode. For example, to find cos 50° and the measure of $\angle A$ when $\cos A = 0.2588$, try the following methods.

Method 1: Enter: 50 [COS]

Display: [0.6427876]

Therefore, cos 50° ≈ 0.6428

Enter: 0.2588 [2nd] [COS^{-1}]

Display: [75.00112969]

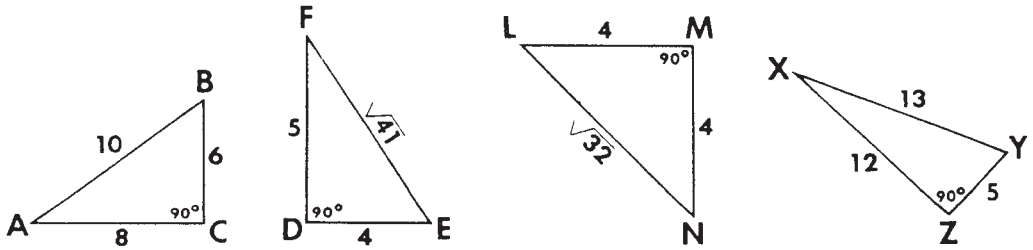
$A \approx 75°$

Method 2: Enter: [COS] 50 [=]

Enter: [2nd] [COS^{-1}] 0.2588 [=]

EXERCISES

In 1–4, represent the cosine of each acute angle.



1. 2. 3. 4.

5. In triangle ABC, $m\angle C = 90$, $AC = 4$, and $BC = 3$. Find cos A.
6. In triangle RST, $m\angle S = 90$, $RS = 5$, and $ST = 12$. Find cos T.

Find the following:

7. cos 21°	**8.** cos 35°	**9.** cos 40°	**10.** cos 45°
11. cos 59°	**12.** cos 67°	**13.** cos 74°	**14.** cos 88°

Find the measure of angle A if:

15. $\cos A = 0.9397$ **16.** $\cos A = 0.6428$ **17.** $\cos A = 0.3584$
18. $\cos A = 0.8910$ **19.** $\cos A = 0.9986$ **20.** $\cos A = 0.0698$

Find the degree measure of angle A to the *nearest degree* if:

21. $\cos A = 0.9750$ **22.** $\cos A = 0.8545$ **23.** $\cos A = 0.6000$
24. $\cos A = 0.5934$ **25.** $\cos A = 0.2968$ **26.** $\cos A = 0.1250$

27. Does the cosine of an angle increase or decrease as the measure of the angle varies from 1 through 89?

28. *a.* Find whether or not cos 80° is twice cos 40°.
b. If the measure of an angle is doubled, is the cosine of the angle also doubled?

29. In triangle ABC, $m\angle C = 90$, $AC = 40$, and $AB = 80$. (*a*) Find cos A to the *nearest ten-thousandth.* (*b*) Find the measure of angle A.

30. In triangle ABC, $m\angle C = 90$, $AC = 12$, and $BC = 5$. (*a*) Find cos B to the *nearest ten-thousandth.* (*b*) Find the degree measure of angle B to the *nearest degree.*

31. Why must the cosine of an acute angle be less than 1?

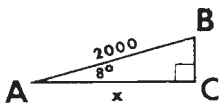
Using the Cosine Ratio to Solve Problems

When the leg adjacent to an acute angle in a right triangle and the hypotenuse of the right triangle are involved in a problem, the use of the cosine ratio will help us find the length of one of these sides when the length of the other side and the measure of the acute angle are given. We proceed as we did when we were using the tangent ratio or the sine ratio.

MODEL PROBLEMS

1. A plane took off from a field and rose at an angle of 8° with the horizontal ground. Find to the *nearest ten feet* the horizontal distance the plane had covered when it had flown 2,000 feet.

Solution: Since the segments mentioned in the problem are the leg adjacent to an acute angle of a right triangle and the hypotenuse of the triangle, use the cosine ratio.



[The solution continues on the next page.]

1. $\cos A = \dfrac{\text{length of leg adjacent to } \angle A}{\text{length of hypotenuse}}$

2. $\cos A = \dfrac{AC}{AB}$ Let $x = AC$.

3. $\cos 8° = \dfrac{x}{2{,}000}$ Cos 8° = 0.9903.

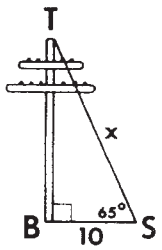
4. $0.9903 = \dfrac{x}{2{,}000}$

5. $x = 2{,}000(0.9903)$

6. $x = 1{,}980.6$

Answer: 1,980 feet correct to the nearest ten feet.

2. A guy wire reaches from the top of a pole to a stake in the ground. The stake is 10 feet from the foot of the pole. The wire makes an angle of 65° with the ground. Find to the *nearest foot* the length of the wire.



Solution: Since the segments mentioned in the problem are the leg adjacent to the acute angle and the hypotenuse of a right triangle, use the cosine ratio.

1. $\cos S = \dfrac{\text{length of leg adjacent to } \angle S}{\text{length of hypotenuse}}$

2. $\cos S = \dfrac{BS}{ST}$ Let $x = ST$.

3. $\cos 65° = \dfrac{10}{x}$ Cos 65° = 0.4226.

4. $0.4226 = \dfrac{10}{x}$

5. $0.4226x = 10$

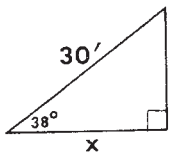
6. $x = \dfrac{10}{0.4226}$

7. $x = 23.6$

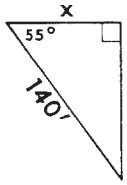
Answer: 24 feet correct to the nearest foot.

EXERCISES

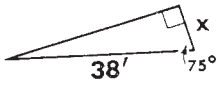
In 1–8, in the given right triangles, find the length of the side marked x to the *nearest foot* or the number of degrees contained in the angle marked x to the *nearest degree*.



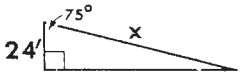
1.



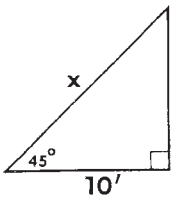
2.



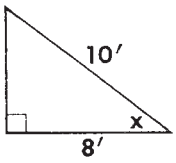
3.



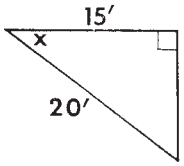
4.



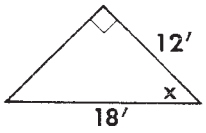
5.



6.



7.



8.

9. A 20-foot ladder leans against a building and makes an angle of 72° with the ground. Find to the *nearest foot* the distance between the foot of the ladder and the building.
10. A man walked 2,500 feet along a straight road which is inclined 12° to the horizontal. Find to the *nearest foot* the horizontal distance traveled by the man.
11. A guy wire attached to the top of a pole reaches a stake in the ground 20 feet from the foot of the pole and makes an angle of 58° with the ground. Find to the *nearest foot* the length of the guy wire.
12. An airplane rises at an angle of 14° with the ground. Find to the *nearest ten feet* the distance it has flown when it has covered a horizontal distance of 1,500 feet.
13. Henry is flying a kite. The kite string makes an angle of 43° with the ground. If Henry is standing 100 feet from a point on the ground directly below the kite, find to the *nearest foot* the length of the kite string.
14. A 30-foot steel girder is leaning against a wall. The foot of the girder is 20 feet from the wall. Find to the *nearest degree* the number of degrees contained in the angle which the girder makes with the ground.

15. A plane took off from an airport. When the plane had flown 4,000 feet in the direction in which it had taken off, it had covered a horizontal distance of 3,900 feet. Find to the *nearest degree* the number of degrees contained in the angle at which the plane rose from the ground.

16. A 40-ft. ladder which is leaning against a wall reaches the wall at a point 36 ft. from the ground. Find to the *nearest degree* the number of degrees contained in the angle which the ladder makes with the wall.

20. Using All Three Trigonometric Ratios

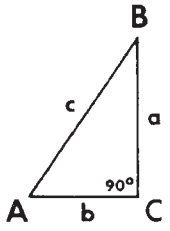
When solving a problem using trigonometry, first make a drawing showing the segments and angles whose measures are given, and the segments and angles whose measures are to be found. Then, in a right triangle, use the proper trigonometric ratios which relate the measures to be found with the measures that are given.

KEEP IN MIND

$$\tan A = \frac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent to } \angle A} = \frac{a}{b}$$

$$\sin A = \frac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}} = \frac{a}{c}$$

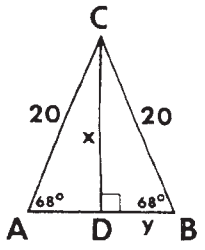
$$\cos A = \frac{\text{length of leg adjacent to } \angle A}{\text{length of hypotenuse}} = \frac{b}{c}$$



MODEL PROBLEMS

1. *Given:* In isosceles triangle ABC, $AC = CB = 20$ and $m\angle A = m\angle B = 68$.
$\overline{CD}$ is an altitude.

To find: *a.* Length of altitude $\overline{CD}$ to the *nearest tenth.*
b. Length of $\overline{AB}$ to the *nearest tenth.*



a.

1. In rt. $\triangle BDC$, $\sin B = \frac{CD}{CB}$.
2. Let $x = CD$. Sin $68° = \frac{x}{20}$
3. $0.9272 = \frac{x}{20}$
4. $x = 20(0.9272)$
5. $x = 18.5440$
6. $x = 18.5$

Answer: $CD = 18.5$ to the nearest tenth.

b.

Since the altitude drawn to the base of an isosceles triangle bisects the base, $AB = 2DB$. Therefore, we will find DB in triangle BDC and double it to find AB.

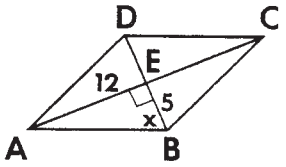
1. In rt. $\triangle BDC$, $\cos B = \frac{DB}{CB}$
2. Let $y = DB$. Cos $68° = \frac{y}{20}$
3. $0.3746 = \frac{y}{20}$
4. $y = 20(0.3746)$
5. $y = 7.4920$
6. $AB = 2y = 2(7.4920)$
7. $AB = 14.9840$
8. $AB = 15.0$

Answer: $AB = 15.0$ to the nearest tenth.

2. The diagonals of a rhombus are 10 and 24. Find, to the *nearest degree*, the number of degrees contained in the angles of the rhombus.

Given: Rhombus $ABCD$.
$BD = 10$.
$AC = 24$.

To find: $m\angle A$, $m\angle B$, $m\angle C$, $m\angle D$.



Solution: Since the diagonals of a rhombus are perpendicular to each other and bisect each other, $\angle AEB$ is a right angle, $EB = \frac{1}{2}(DB) = \frac{1}{2}(10) = 5$, and $AE = \frac{1}{2}(AC) = \frac{1}{2}(24) = 12$.

Since in rhombus $ABCD$, diagonal $\overline{DB}$ bisects $\angle ABC$, then $m\angle ABC = 2(m\angle ABE)$. Therefore, we will find the measure of $\angle ABE$ in right triangle AEB and multiply the result by 2 to find $m\angle ABC$. Let us represent the measure of $\angle ABE$ by x.

[The solution continues on the next page.]

1. In rt. $\triangle AEB$, $\tan x = \dfrac{\text{length of leg opposite } \angle ABE}{\text{length of leg adjacent to } \angle ABE}$

2. $\tan x = \frac{12}{5}$ Express $\frac{12}{5}$ as the decimal 2.4000.

3. $\tan x = 2.4000$

4. $x = 67$

5. $m\angle ABC = 2x = 2(67) = 134$

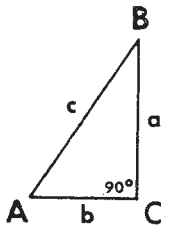
Since the consecutive angles of a rhombus are supplementary, $\angle BCD$ is supplementary to $\angle ABC$. Therefore, $m\angle BCD = 180 - 134 = 46$.

Since the opposite angles of a rhombus are congruent, $m\angle CDA = m\angle ABC = 134$ and $m\angle DAB = m\angle BCD = 46$.

Answer: The angles of the rhombus contain 46°, 134°, 46°, 134°.

EXERCISES

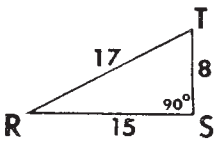
Exercises 1–7 refer to rt. $\triangle ABC$. Name the ratio that can be used to find:



Ex. 1–7

1. the measure of side a when the measures of angle A and side c are given.
2. the measure of side b when the measures of angle A and side c are given.
3. the measure of side c when the measures of side a and angle A are given.
4. the measure of side b when the measures of side a and angle B are given.
5. the measure of angle A when the measures of side a and side b are given.
6. the measure of angle B when the measures of side a and side c are given.
7. the measure of side a when the measures of side b and angle A are given.

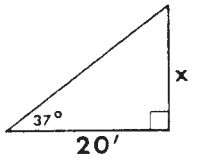
Exercises 8–13 refer to $\triangle RST$. In each exercise, give the value of the ratio as a fraction.



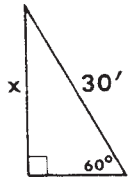
Ex. 8–13

8. sin R
9. tan T
10. sin T
11. cos R
12. cos T
13. tan R

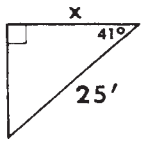
In 14–21, in the given right triangles, find the length of the side marked x to the *nearest foot* or the number of degrees contained in the angle marked x to the *nearest degree*.



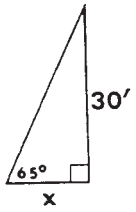
14.



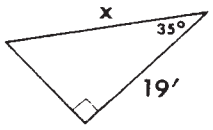
15.



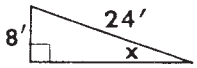
16.



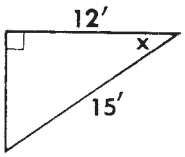
17.



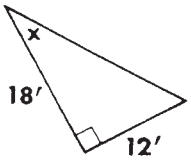
18.



19.



20.



21.

22. If $\cos A = \sin 30°$, angle A contains _____ degrees.
23. Diameter $\overline{AB}$ of a circle is 13 and chord $\overline{AC}$ is 12. Chord $\overline{CB}$ is drawn. Find the value of $\sin B$.
24. If in right triangle ACB $m\angle C = 90$, $m\angle A = 66$, and $AC = 100$, then BC to the *nearest integer* is _____.
25. In right triangle ABC, $m\angle C = 90$, $m\angle B = 28$, and $BC = 30$ ft. Find AB to the *nearest foot.*
26. In triangle ABC, $m\angle C = 90$, $\tan A = .7$, and $AC = 40$. Find BC.
27. In triangle ABC, $m\angle A = 42$, $AB = 14$, and $\overline{BD}$ is the altitude to $\overline{AC}$. Find BD to the *nearest tenth.*
28. In triangle ABC, $\overline{AC} \cong \overline{BC}$, $m\angle A = 50$, and $AB = 30$. Find to the *nearest tenth* the length of the altitude from vertex C.
29. ABC is an isosceles triangle with $\overline{AB} \cong \overline{AC}$. If angle B contains 35° and $BC = 20$, the altitude upon $\overline{BC}$ to the *nearest integer* is _____.
30. The longer side of a rectangle is 10 and a diagonal makes an angle of 27° with this side. Find to the *nearest integer* the shorter side of the rectangle.
31. At a point on the ground 100 feet from the foot of a flagpole, the angle of elevation of the top of the pole contains 31°. The height of the flagpole to the *nearest foot* is _____.
32. Find to the *nearest foot* the height of a church spire that casts a shadow of 50 feet when the angle of elevation of the sun contains 68°.
33. From the top of a lighthouse 190 feet high, the angle of depression of a boat out at sea contains 34°. Find to the *nearest foot* the distance from the boat to the foot of the lighthouse, the foot of the lighthouse being at sea level.
34. In triangle ABC, $m\angle C = 90$, $AB = 30$, and $BC = 15$. How many degrees are contained in angle A?

35. The legs of a right triangle are 3 and 4. Find the degree measure of the smallest angle of this triangle to the *nearest degree.*

36. The length of the hypotenuse $\overline{AB}$ of right triangle ABC is twice the length of leg $\overline{BC}$. Find the number of degrees in angle ABC.

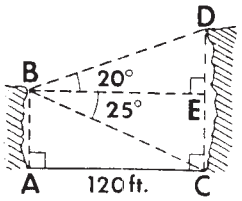
37. In rectangle $ABCD$, diagonal $\overline{AC}$ is 11 and side $\overline{AB}$ is 7. Find the degree measure of angle CAB to the *nearest degree.*

38. Find to the *nearest degree* the degree measure of the angle of elevation of the sun if a post 5 feet high casts a shadow 10 feet long.

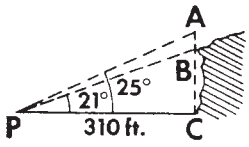
39. $\overline{CD}$ is the altitude on the hypotenuse of right triangle ABC. $AB = 25$ and $AC = 20$. Find the length of segment $\overline{BD}$, the length of altitude $\overline{CD}$, and the degree measure of angle B to the *nearest degree.*

40. The longer diagonal of a rhombus is 24 ft. and the shorter diagonal is 10 ft. (*a*) Find the perimeter of the rhombus. (*b*) Find to the *nearest degree* the degree measure of the angle which the longer diagonal makes with a side of the rhombus.

41. The altitude on the hypotenuse of a right triangle divides the hypotenuse into segments whose measures are 9 and 4. Find to the *nearest degree* the degree measure of the smaller acute angle of the original triangle.



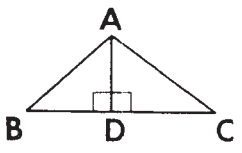
Ex. 42



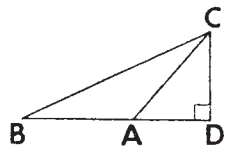
Ex. 43

42. $\overline{AB}$ and $\overline{CD}$ represent cliffs on opposite sides of a river 120 feet wide. From B, the angle of elevation of D contains 20° and the angle of depression of C contains 25°. Find to the *nearest foot:* (*a*) the height of the cliff represented by $\overline{AB}$. (*b*) the height of the cliff represented by $\overline{CD}$.

43. In the diagram, P represents a point 310 feet from the foot of a vertical cliff $\overline{BC}$. $\overline{AB}$ represents a flagpole standing on the edge of the cliff. At P, the angle of elevation of B contains 21° and the angle of elevation of A contains 25°. Find to the *nearest foot:* (*a*) the distance AC. (*b*) AB, which represents the length of the flagpole.



Ex. 44



Ex. 45

44. In triangle ABC, $AB = 30$ feet, $m\angle B = 42$, $m\angle C = 36$, and $\overline{AD}$ is an altitude. (*a*) Find to the *nearest foot* the length of $\overline{AD}$. (*b*) Using the result obtained in part *a*, find to the *nearest foot* the length of $\overline{DC}$.

45. CD represents the height of a building. $AD = 85$ ft. and $m\angle D = 90$. At A, the angle of elevation of the top of the building, $\angle CAD$ contains 49°. At B, the angle of elevation of the top of the building, $\angle CBD$, contains 26°. (*a*) Find the height of the building, CD, to the *nearest foot.* (*b*) Find CB, the distance from the top of the building to B, to the *nearest foot.*

46. Angle D in quadrilateral $ABCD$ is a right angle, and diagonal $\overline{AC}$ is perpendicular to $\overline{BC}$. $BC = 20$, $m\angle B = 35$, and $m\angle DAC = 65$. (*a*) Find AC to the *nearest integer.* (*b*) Using the result obtained in answer to *a*, find DC to the *nearest integer.*

47. The diagonals of a rectangle are each 22 and intersect at an angle of 110°. Find to the *nearest integer* the sides of the rectangle.

48. In rhombus $ABCD$, the measure of diagonal $\overline{AC}$ is 80 and $m\angle BAC = 42$. (*a*) Find the length of a side of the rhombus to the *nearest integer.* (*b*) Find the length of an altitude of the rhombus to the *nearest integer.*

49. In circle O, chord $\overline{AC}$ forms an angle of 37° with diameter $\overline{AB}$. If the length of $\overline{AB}$ is 20 inches, find to the *nearest inch:* (*a*) the length of the chord $\overline{BC}$. (*b*) the distance from the center of the circle to chord $\overline{AC}$.

50. A circle is inscribed in a regular pentagon. If a side of the pentagon measures 50, find to the *nearest integer* the length of the radius of the circle.

51. $\overline{PA}$ and $\overline{PB}$ are tangents to circle O and intersect at an angle of 60°. Radius $\overline{OA}$ is 8 inches long. Find to the *nearest inch* the length of $\overline{AP}$.

52. In right triangle ABC, the length of hypotenuse $\overline{AB}$ is 100 and $m\angle A = 18$. (*a*) Find AC and BC to the *nearest integer.* (*b*) Show that the results obtained in answer to *a* are approximately correct by using the relationship $(AB)^2 = (AC)^2 + (BC)^2$.

21. Completion Exercises

Write a word or expression that, when inserted in the blank, will make the resulting statement true.

1. If two polygons are congruent, they must be _____.
2. Two polygons are similar if their corresponding angles are congruent and the ratios of their corresponding sides are _____.
3. The segments of one of two chords intersecting within a circle measure r and s. If the length of one segment of the other chord is m, the length of the other segment of that chord in terms of r, s, and m is _____.
4. If two triangles are similar, their corresponding _____ are congruent.
5. If the measure of one acute angle of a right triangle is 30, the ratio of the length of the shorter leg to the length of the hypotenuse is _____.

6. In a right triangle, the cosine of an acute angle is the ratio of the length of the _____ side to the length of the hypotenuse.
7. Triangle ABC is a right triangle with its right angle at C, and E is any point on side $\overline{AC}$. Segment $\overline{ED}$ is perpendicular to $\overline{AB}$ at D. Complete the proportion $DE{:}CB = DA{:}$_____.
8. The formula for the length of the diagonal of a square, d, in terms of the length of its side, s, is $d =$ _____.
9. If the length of the hypotenuse of a right triangle is twice the length of the shorter leg, the smallest angle of the triangle contains _____ degrees.
10. The formula for the length of the altitude of an equilateral triangle, h, in terms of the length of its side, s, is $h =$ _____.
11. The line segment whose endpoints are the midpoints of two sides of a triangle is _____ to the third side.
12. The perimeters of two _____ polygons are to each other as the lengths of any pair of corresponding sides.
13. Two secants are drawn to a circle from an outside point. The length of the first secant is a and the length of its external segment is b. If the length of the second secant is c, the length of its external segment in terms of a, b, and c is _____.
14. In trapezoid $ABCD$ with bases $\overline{AB}$ and $\overline{DC}$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. Triangle DEC is similar to triangle _____.
15. In triangle ABC, E is the midpoint of $\overline{AC}$ and D is the midpoint of $\overline{BC}$. The ratio of $ED{:}AB$ is _____.
16. In right triangle ABC, $\overline{BD}$ is the altitude to the hypotenuse $\overline{AC}$. Triangle ABD is similar to triangle _____.
17. In a right triangle whose acute angles contain 30° and 60°, if the length of the side opposite the 30° angle is m, then the length of the side opposite the 60° angle is _____.
18. In an isosceles right triangle, if the length of each of the congruent sides is represented by a, then the length of the hypotenuse is represented by _____.
19. If the sides of a triangle measure 10, 24, and 26, it is a(an) _____ triangle.
20. In $\triangle ABC$, D, E, and F are the midpoints of sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ respectively. If $\overline{DE}$, $\overline{EF}$ and $\overline{FD}$ are drawn, then $\triangle EFD$ is _____ to $\triangle ABC$.

22. True-False Exercises

If the statement is always true, write *true;* if the statement is not always true, write *false.*

1. If the angles of one polygon are congruent respectively to the angles of a second polygon, the polygons are similar.
2. Isosceles right triangles are similar.
3. If the lengths of the sides of two polygons are in proportion, the polygons are similar.
4. The diagonals of a trapezoid divide each other proportionally.
5. A line which contains the midpoints of two sides of a triangle cuts off a triangle similar to the given triangle.
6. The bisector of an angle of a triangle divides the triangle into two similar triangles.
7. If the acute angles of a right triangle measure 30° and 60°, the lengths of the legs opposite these angles are in the ratio 1:2.
8. Isosceles triangles are similar if their vertex angles are congruent.
9. If the measure of acute angle A is twice the measure of acute angle B, then $\sin A$ is twice $\sin B$.
10. As the measure of an acute angle increases, the cosine of the angle increases.
11. The length of the hypotenuse of an isosceles right triangle is equal to the length of a leg multiplied by $\sqrt{2}$.
12. If an angle of a rhombus measures 60°, the shorter diagonal is congruent to a side of the rhombus.
13. If the altitude is drawn to the hypotenuse of a right triangle, the length of a leg is the geometric mean between the lengths of the segments of the hypotenuse.
14. If a set of chords is drawn through a point inside a circle, the product of the lengths of the segments of each chord is constant.
15. If the length of each side of a triangle is tripled, the measure of each angle is also tripled.
16. Congruent triangles are similar.
17. The lengths of two corresponding altitudes in two similar triangles have the same ratio as the lengths of any pair of corresponding sides.

18. In right triangle ABC if $\overline{CD}$ is the altitude upon hypotenuse $\overline{AB}$, then $(CD)^2 = AD \times DB$.
19. If $\overline{CD}$ is a diameter of a circle and B is any point on the circle, then $(CB)^2 + (BD)^2 = (CD)^2$.
20. If, from point P outside a circle, tangent $\overline{PA}$ and secant $\overline{PD}$, which intersects the circle at C, are drawn, then $(PA)^2 = PC \times CD$.

23. "Always, Sometimes, Never" Exercises

If the blank space in each of the following exercises is replaced by the word *always, sometimes,* or *never,* the resulting statement will be true. Select the word which will correctly complete each statement.

1. If the angles of one polygon are respectively congruent to the angles of another polygon, the polygons are _____ similar.
2. Congruent polygons are _____ similar.
3. Similar triangles are _____ congruent triangles.
4. If in two polygons, the lengths of corresponding sides are in proportion, the polygons are _____ similar.
5. $\overline{DE}$ and $\overline{D'E'}$ are bases of isosceles triangles DEF and $D'E'F'$. If $DE{:}D'E' = DF{:}D'F'$, the triangles are _____ similar.
6. The length of side $\overline{AB}$ of triangle ABC is 16, and a line intersects sides $\overline{AC}$ and $\overline{BC}$ in D and E respectively. If DE equals 8, then $\overline{DE}$ is _____ parallel to $\overline{AB}$.
7. If two chords of a circle intersect, the product of the lengths of the segments of one chord is _____ equal to the product of the lengths of the segments of the other chord.
8. The sine of an angle is _____ equal to the cosine of that angle.
9. If a vertex angle of one isosceles triangle is congruent to the corresponding angle of another isosceles triangle, the two triangles are _____ similar.
10. A median in a triangle _____ divides the triangle into two similar triangles.
11. If two angles of a triangle measure 30° and 60°, the length of the side opposite the 60° angle is _____ twice the length of the side opposite the 30° angle.
12. The sine of an acute angle _____ equals the cosine of the complementary angle.
13. As the measure of an acute angle increases, the tangent of the angle _____ decreases.

14. The diagonals $\overline{AC}$ and $\overline{BD}$ of quadrilateral $ABCD$ inscribed in a circle intersect at E. Triangle AED is _____ similar to triangle BEC.
15. In a right triangle, the length of the altitude drawn to the hypotenuse is _____ the geometric mean between the lengths of the segments of the hypotenuse.
16. A rhombus whose side measures 8 inches _____ has a diagonal 16 inches long.
17. If the lengths of two sides of a triangle are proportional to the lengths of two sides of another triangle, the triangles are _____ similar.
18. If the sides of one triangle are parallel to the sides of another triangle, the triangles are _____ similar.
19. A diagonal of a rhombus is _____ congruent to a side of the rhombus.
20. Two right triangles are _____ similar if the lengths of the legs of one triangle are proportional to the lengths of the legs of the other triangle.

24. Multiple-Choice Exercises

Write the letter preceding the word or expression that best completes the statement.

1. Two figures must be similar if they are (*a*) rectangles (*b*) equilateral triangles (*c*) rhombuses.
2. If altitude $\overline{CD}$ is drawn upon the hypotenuse $\overline{AB}$ of right triangle ABC, then $(AC)^2$ equals (*a*) $CB \times DB$ (*b*) $AB \times AD$ (*c*) $DB \times AD$.
3. If in right triangle ABC, $\overline{AB}$ is the hypotenuse and $\overline{CD}$ is the altitude to the hypotenuse, then (*a*) $(CD)^2 = AD \times DB$ (*b*) $(CD)^2 = AB \times AD$ (*c*) $(CD)^2 = AC \times CB$.
4. If the lengths of the segments of one of two chords intersecting within a circle are represented by r and s and the lengths of the segments of the other chord are represented by v and w, then (*a*) $r \times s = v \times w$ (*b*) $r + s = v + w$ (*c*) $\frac{r}{s} = \frac{v}{w}$.
5. If from a point outside a circle a tangent and a secant are drawn to the circle, the length of the tangent is the geometric mean between (*a*) the length of the whole secant and the length of its internal segment (*b*) the lengths of the external and internal segments of the secant (*c*) the length of the whole secant and the length of its external segment.
6. If in right triangle ABC $m\angle A = 30$ and $m\angle B = 60$, then (*a*) $AC = 2BC$ (*b*) $AC = \frac{1}{2}AB$ (*c*) $AC = BC\sqrt{3}$.
7. If the altitude $\overline{CD}$ is drawn to the hypotenuse $\overline{AB}$ of right triangle ABC, then AC is the geometric mean between (*a*) AD and DB (*b*) AB and AD (*c*) AB and BC.

8. The length of the diagonal of a square, d, is equal to the length of a side, s, multiplied by (*a*) $\sqrt{2}$ (*b*) $\sqrt{3}$ (*c*) 2.
9. If the length of each side of a triangle is multiplied by 2, then the measure of each angle (*a*) is multiplied by 2 (*b*) is multiplied by 4 (*c*) remains unchanged.
10. Two chords intersect inside a circle. The lengths of the segments of one chord are 6 and 12, and the lengths of the segments of the other chord are represented by x and $x + 1$. An equation that can be used to find x is (*a*) $\frac{x+1}{x} = \frac{6}{12}$ (*b*) $2x + 1 = 18$ (*c*) $x^2 + x = 72$.
11. If an altitude of an equilateral triangle is $5\sqrt{3}$, the length of a side is (*a*) 10 (*b*) 5 (*c*) $10\sqrt{3}$.
12. The acute angles of a right triangle contain 30° and 60° respectively. The lengths of the legs of the triangle are in the ratio (*a*) 1:2 (*b*) $1:\sqrt{2}$ (*c*) $1:\sqrt{3}$.
13. A tangent and a secant are drawn to a circle from an external point. The external segment of the secant is 6 and the internal segment is 5. The length of the tangent is (*a*) $\sqrt{30}$ (*b*) $\sqrt{55}$ (*c*) $\sqrt{66}$.
14. If the lengths of the sides of a triangle are 3, 4, and 5 respectively, the value of the sine of the smallest angle is (*a*) $\frac{3}{5}$ (*b*) $\frac{3}{4}$ (*c*) $\frac{4}{5}$.
15. In triangle ABC, $m\angle C = 90$. If $AB = 15$ and $AC = 12$, which of the following statements is *not* true? (*a*) $\tan A = \frac{3}{4}$ (*b*) $\sin B < 1$ (*c*) $\tan B < 1$.
16. If in triangle ABC, $\overline{CD}$ is the altitude upon $\overline{AB}$, then CD equals (*a*) $AD \times \sin A$ (*b*) $AD \times \cos A$ (*c*) $AD \times \tan A$.
17. If the three sides of a triangle measure 21, 28, and 35, the triangle is (*a*) right (*b*) obtuse (*c*) acute.
18. In an isosceles triangle whose vertex angle contains 72°, if the length of the altitude drawn to the base is m, then the length of the base of the triangle is (*a*) $2m \cos 36°$ (*b*) $2m \sin 36°$ (*c*) $2m \tan 36°$.
19. If in triangle ABC, $\overline{BD}$ is the altitude upon $\overline{AC}$, then AD equals (*a*) $AB \times \sin A$ (*b*) $AB \times \tan A$ (*c*) $AB \times \cos A$.
20. If the ratio of the lengths of a pair of corresponding sides in two similar triangles is 4:1, the ratio of the lengths of a pair of corresponding altitudes is (*a*) 2:1 (*b*) 4:1 (*c*) 16:1.

25. Construction Exercises

The basic constructions involved in the following exercises, which are to be done with straightedge and compasses, appear in Chapter 13, which begins on page 614.

1. Divide line segment $\overline{AB}$ into two parts whose lengths shall be proportional to the lengths of the two given segments m and n.

Ex. 1

2. Construct the fourth proportional to the three line segments a, b, and c.

Ex. 2

3. Given the line segments a, b, and c, construct line segment x such that $a:b = c:x$.

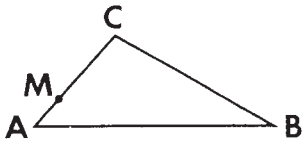
Ex. 3

4. Divide line segment $\overline{AB}$ into three parts that are equal in length.

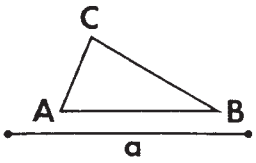
Ex. 4

5. Construct the geometric mean between line segments a and b.

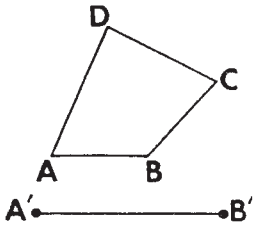
Ex. 5



Ex. 6



Ex. 7



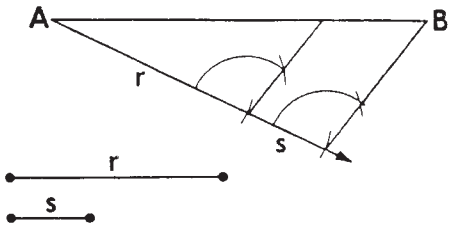
Ex. 8

6. Through point M, construct a line which will divide sides $\overline{AC}$ and $\overline{BC}$ of triangle ABC proportionally.
7. On line segment a corresponding to side $\overline{AB}$ of triangle ABC, construct a triangle similar to triangle ABC.
8. On line segment $\overline{A'B'}$ corresponding to side $\overline{AB}$ in polygon $ABCD$, construct a polygon similar to polygon $ABCD$.
9. If a, b, and c are given line segments, construct a line segment x such that:
   a. $a:b = x:c$ *b.* $a:x = c:b$ *c.* $x:a = b:c$ *d.* $a:2b = c:x$
10. If a, b, and c are given line segments, construct a line segment x such that:
    a. $x = \frac{ab}{c}$ *b.* $x = \frac{ac}{b}$ *c.* $x = \frac{2bc}{a}$ *d.* $x = \frac{2ab}{3c}$
11. If a and b are given line segments, construct a line segment x such that:
    a. $x = \frac{a^2}{b}$ *b.* $x = \frac{b^2}{a}$ *c.* $x = \frac{2a^2}{b}$ *d.* $x = \frac{b^2}{3a}$
12. Divide a given line segment $\overline{AB}$ into two segments whose lengths are in the ratio:
    a. 1:2 *b.* 1:3 *c.* 2:3 *d.* 3:4

13. Divide a given line segment $\overline{AB}$ into three segments whose lengths are in the ratio 1:3:4.

14. If a and b are given line segments, construct line segment x such that:
$a.\ a:x = x:b$ $\quad b.\ 2a:x = x:b$ $\quad c.\ x = \sqrt{ab}$ $\quad d.\ x = \sqrt{3ab}$

15. The diagram shows the division of given line segment $\overline{AB}$ into two segments whose lengths are in the ratio $r:s$.



Ex. 15

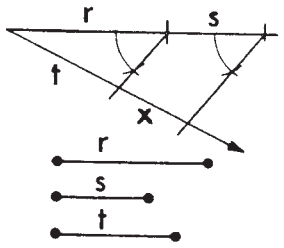
Which statement, a, b, or c, is used to prove that the construction is correct?

a. A line that divides two sides of a triangle proportionally is parallel to the third side.

b. If a line is drawn which intersects two sides of a triangle in two different points and is parallel to the third side, the line divides those sides proportionally.

c. If two triangles are similar, the lengths of their corresponding sides are in proportion.

16. The diagram shows the construction of the fourth proportional to three given line segments, r, s, and t. Which statement, a, b, or c, is used to prove that the construction is correct?



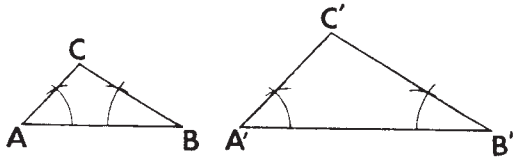
Ex. 16

a. If two triangles are similar, the corresponding sides are in proportion.

b. If a line divides two sides of a triangle proportionally, the line is parallel to the third side.

c. If a line is parallel to one side of a triangle and intersects the other two sides in two different points, the line divides the other two sides proportionally.

17. The diagram shows the construction of a triangle similar to a given triangle on a given line segment as a base. Which statement, a, b, or c, is used to prove that the construction is correct?



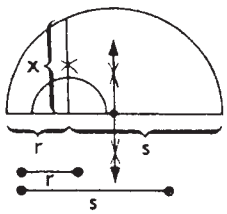
Ex. 17

a. If two triangles are congruent, they are similar.

b. Two triangles are similar if two angles of one are congruent respectively to two angles of the other.

c. If two triangles are similar, the angles of one triangle are congruent to the corresponding angles of the other.

18. The diagram shows the construction of the geometric mean between two given line segments. Which statement, *a, b,* or *c,* is used to prove that the construction is correct?



Ex. 18

a. The length of the altitude drawn to the hypotenuse of a right triangle is the geometric mean between the lengths of the segments of the hypotenuse.

b. If from any point on the arc of a chord a line segment is drawn perpendicular to the chord and terminating in the chord, the length of the perpendicular is the geometric mean between the lengths of the segments of the chord.

c. The length of a leg of a right triangle is the geometric mean between the length of the hypotenuse and the length of the projection of the leg on the hypotenuse.