

CHAPTER VII



Perimeters and Areas of Polygons

In earlier mathematics courses, you have no doubt learned how to compute the perimeters and areas of some geometric figures by using special formulas. Hence, you have some understanding of the meanings of the words *perimeter* and *area*. In this chapter, we will make a formal study of perimeters and areas as part of our deductive system.

1. Finding the Perimeter of a Polygon

Evaluating Perimeter Formulas

To find the perimeter of a polygon, we add the lengths of its sides. For example, to find the perimeter P of a square whose side is s , we use the formula $P = s + s + s + s$ or $P = 4s$. In each case, we are evaluating an algebraic expression, $P = s + s + s + s$ or $P = 4s$, to find the perimeter P .

When finding the perimeter of a figure, all lengths should be expressed in the same unit of measure.

MODEL PROBLEMS ~~~~~

1. Find the perimeter of each polygon.
 - a. A triangle whose sides measure 3 meters, 400 centimeters, and 5 meters.
 - b. A parallelogram whose two consecutive sides measure 5 feet and 6 feet.
 - c. A trapezoid whose sides measure 5 centimeters, 500 millimeters and bases that measure 6 centimeters and 14 centimeters.

Solution:

- a.
 1. Use the formula $P = s + s + s$
 2. Change 400 centimeters to 4 meters. (Recall that $100\text{ cm} = 1\text{ m}$)
 3. $P = 3 + 4 + 5$
 4. $P = 12$

Answer: The perimeter of the triangle is 12 meters.

- b. 1. Since we know that opposite sides of a parallelogram are congruent, we can use the formula $P = 2s_1 + 2s_2$ to find the perimeter of the parallelogram.
2. Let $s_1 = 5$ feet and $s_2 = 6$ feet.
3. $P = 2(5) + 2(6)$
4. $P = 10 + 12$
5. $P = 22$

Answer: The perimeter of the parallelogram is 22 feet.

- c. 1. Use the formula $P = s + s + s + s$
2. Change 500 millimeters to 5 centimeters (Recall that 100 mm = 1 cm)
3. $P = 5 + 5 + 6 + 14$
4. $P = 30$

Answer: The perimeter of the trapezoid is 30 centimeters.

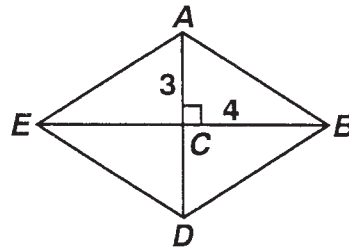
2. Find the perimeter of a rhombus with diagonals 6 meters and 8 meters.

Solution:

1. Let s = the length of a side.
2. The diagonals AD and BE are perpendicular.

In the right triangle ACB ,

3. $3^2 + 4^2 = s^2$ (Pythagorean Theorem)
4. $9 + 16 = s^2$
5. $s^2 = 25$
6. $s = \sqrt{25}$
7. $s = 5$
8. A rhombus has four congruent sides.
9. $P = 4s$
10. $P = 4(5)$
11. $P = 20$



Answer: The perimeter of the rhombus is 20 meters.

Finding the Perimeter of a Regular Polygon

A *regular polygon* is a polygon in which all of the sides have equal measures and all of the angles have equal measures. If we know the length of one side of a regular polygon, we can find the perimeter of that polygon. For example, find the perimeter of an equilateral triangle with a side that measures 4 inches. Since an equilateral triangle is a polygon in which all three sides have equal measures, the perimeter of the equilateral triangle is 3s

where s is the length of a side. Therefore, the perimeter of an equilateral triangle with a side that measures 4 inches is $3 \cdot 4 = 12$ inches.

To find the perimeter of a regular polygon, we need to know the length of one of the sides and the number of sides the regular polygon has. Therefore, the perimeter of a regular polygon is equal to the length of a side times the number of sides.

MODEL PROBLEMS

1. Find the perimeter of a regular octagon with a side that measures 5 feet.

Solution:

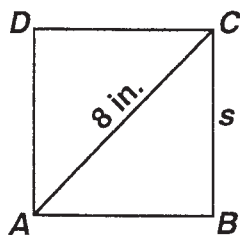
An octagon has eight sides. Let P = the perimeter, s = the length of one side, and n = the number of sides.

1. $P = sn$
2. $P = 5(8)$
3. $P = 40$

Answer: The perimeter of the regular octagon is 40 feet.

2. Find to the *nearest tenth of an inch* the perimeter of a square whose diagonal measures 8 inches.

Solution:



1. Let s = the length of a side and n = the number of sides of the square.

In the right triangle ABC ,

2. $s^2 + s^2 = 8^2$ (Pythagorean Theorem)
3. $2s^2 = 64$
4. $s^2 = 32$
5. $s = \sqrt{32}$
6. A square has four sides.
7. $P = sn$
8. $P = \sqrt{32}(4)$
9. $P = 22.6$

Answer: The perimeter of the square is 22.6 inches, to the nearest tenth of an inch.

3. Find the length of a side of a regular pentagon with a perimeter of 30 centimeters.

Solution:

A pentagon has five sides.

Let P = the perimeter, s = the length of a side, and n = the number of sides.

1. $P = sn$
2. $30 = s(5)$
3. $s = 6$

Answer: The length of a side of the regular pentagon is 6 centimeters.

EXERCISES

In 1–6, find the perimeter of each polygon.

1. A triangle whose sides measure 2 meters, 200 centimeters, and 20,000 millimeters
2. A parallelogram with two consecutive sides measuring 5 units and 7 units
3. A right triangle whose legs measure 5 inches and 12 inches
4. A trapezoid whose sides measure 2 inches each and bases that measure 1 inch and 3 inches
5. A rhombus with diagonals 24 inches and 32 inches
6. A rectangle with two consecutive sides that measure 6 inches and 1 foot

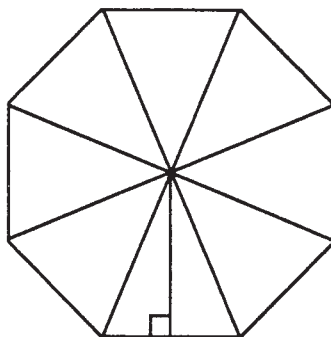
In 7–11, find the perimeter of each regular polygon.

7. A square with the length of a side equal to 2 feet
8. A hexagon with the length of a side equal to 3 inches
9. A square with the length of a side equal to $4a$
10. A decagon (10-sides) with the length of a side equal to $(x + 3)$
11. A triangle with one side equal to $\frac{1}{2}x$ units

In 12–16, find the length of a side of each regular polygon.

12. A square with a diagonal equal to $\sqrt{2}$ units
13. An octagon with a perimeter of 5 centimeters
14. A hexagon with a perimeter of 1 meter
15. A pentagon with a perimeter of 10 feet
16. A square with a diagonal equal to $4x$ units
17. If a regular hexagon has a perimeter of $36(x + 8)$, what is the length of one side of the regular hexagon?
18. Find the perimeter of a rhombus if the lengths of its diagonals are 12 inches and 16 inches.
19. Find the length of the diagonal of a square with a perimeter of 4 feet.

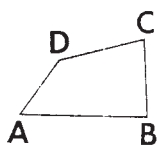
20. The diagonals of a regular octagon bisect each other. Use the figure below to find the perimeter of a regular octagon with a diagonal of 4 inches and an interior angle of 135° .



2. Understanding the Meaning of Area

We have learned that every simple closed polygon has an interior region. The points of the interior region are not points of the polygon. The term *polygonal region* is used to refer to the union of the polygon and its interior. In Fig. 7-1 are pictured polygon $ABCD$ and polygonal region $ABCD$.

Polygon $ABCD$



Polygonal Region $ABCD$

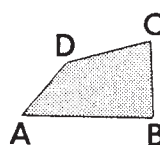


Fig. 7-1

When we measure a line segment, we determine a number which results from discovering how many times a certain unit length is contained in the line segment. If the length of a line segment is 7 when the unit of measure is the inch, the line segment contains the linear unit, the inch, 7 times.

When we measure an angle, we determine a number which results from discovering how many times a certain unit angle is contained in the angle. If the measure of an angle is 30 when the unit of measure is the degree, the angle contains the unit angle, the degree, 30 times.

Similarly, to measure the area of a polygonal region, we determine a number which results from discovering how many times a certain unit of area is contained in the region. There are many such possible units of measure. However, you know from past experience that the *unit square* is a convenient unit to use for measuring the size of a polygonal region. If we first select a unit of length and then draw a square whose side is that unit (Fig. 7-2), the area of the square is a unit of area. For example, if the length of a side of the unit square is 1 inch, the area

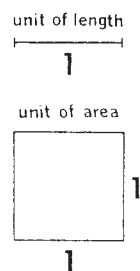


Fig. 7-2

of the square is a unit of area called a *square inch*. If the measure of the area of a polygonal region is 100 when the unit of measure is a square the length of each of whose sides is 1 inch, the polygonal region contains the area unit, the square inch, 100 times.

Definition. The area of a polygonal region is the number of area units contained within the region.

In the future, in order to simplify the language that we use to express ourselves, we shall talk about *finding the area of a polygon* rather than the area of a polygonal region. For example, we shall be finding the area of a triangle rather than the area of a triangular region.

From your experience with congruent triangles, the following postulate should be reasonable to you.

Postulate 42. If two triangles are congruent, they have the same area.

In Fig. 7-3, if $\triangle ABC \cong \triangle A'B'C'$, then the area of $\triangle ABC = \text{area of } \triangle A'B'C'$.

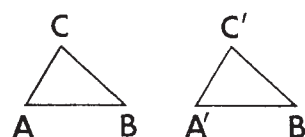


Fig. 7-3

Addition and Subtraction of Areas

Postulate 43. If a polygon which encloses a region is separated into several polygons which do not overlap, its area is the sum of the areas of these polygons. (Area-Addition Postulate)

In Fig. 7-4, \overline{AC} separates polygon $ABCD$ into the two triangles ABC and CDA , which do not overlap. Hence, area of polygon $ABCD = \text{area of } \triangle ABC + \text{area of } \triangle CDA$. It follows from this statement that:

1. Area of polygon $ABCD - \text{area of } \triangle ABC = \text{area of } \triangle CDA$.
2. Area of polygon $ABCD - \text{area of } \triangle CDA = \text{area of } \triangle ABC$.

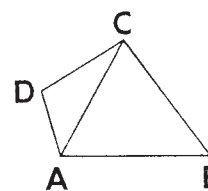


Fig. 7-4

It is often difficult or impossible to find the area of a polygon by counting the number of units of area it contains. For this reason, mathematicians have developed formulas which can be used conveniently to find the areas of familiar geometric figures. We shall soon study some of these formulas.

EXERCISES

In 1-10, answer *yes* or *no*.

1. Does every polygon have an area?
2. Does an angle have an area?
3. Does a line segment have an area?
4. Can a triangle have an area of 10 inches?

5. If two triangles are congruent, must they have the same area?
6. If two triangles have the same area, must they be congruent?
7. If two triangles do not have the same area, can they be congruent?
8. Can a triangle and a square have the same area?
9. Does the diagonal of a parallelogram divide it into two triangles whose areas are equal?
10. Must a plane figure be a polygon in order to have an area?
11. A square each of whose sides is 1 inch long is contained in a rectangle exactly fifty times. Find the area of the rectangle.
12. If the diagonals of a square are drawn, what is the relationship of the areas of the four triangles that are formed?
13. If two equilateral triangles have equal perimeters, what is the relationship of their areas?

3. Finding the Area of a Rectangle

Fig. 7-5 represents a rectangle the length of whose base is 6 inches and whose altitude is 3 inches. The horizontal and vertical segments shown form unit areas of 1 square inch. By counting the number of unit squares contained in the rectangle, we find that the area enclosed in the rectangle is 18 square inches. Notice, too, that the rectangle contains 3 rows of squares, every row containing 6 squares, each of which is 1 square inch. Therefore, the area of the rectangle is 3×6 , or 18 square inches.

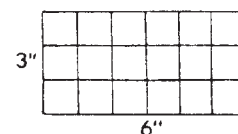


Fig. 7-5

The preceding example illustrates the reasonableness of accepting the following postulate, provided that the length of the base and the length of the altitude of a rectangle are expressed in the same linear unit:

Postulate 44. The area of a rectangle is equal to the product of the length of its base and the length of its altitude.

In Fig. 7-6, the area, A , of rectangle $ABCD$, the length of whose base is represented by b and the length of whose altitude is represented by h , is given by the formula $A = bh$.

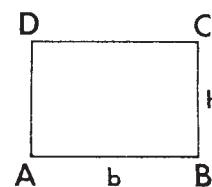


Fig. 7-6

NOTE. In the statement of postulate 44, we might have talked about the “base” and the “altitude” of the rectangle rather than the “length of the base” and the “length of the altitude.” From time to time, we will use this simplified language. Remember that “base” and “altitude” refer to lengths which we know are numbers.

We will also use simplified language when we deal with other types of quadrilaterals and with triangles. For example, instead of saying that the length of the base of a parallelogram is 10 feet and the length of the altitude drawn to that base is 4 feet, we may simply say that the base of the parallelogram is 10 feet and the altitude drawn to that base is 4 feet.

MODEL PROBLEMS

1. Find the area of a rectangle whose base is $1\frac{1}{2}$ feet and whose altitude is 6 inches.

Solution: Since the base and the altitude of the rectangle are expressed in terms of different linear units, we must first convert them to the same linear unit before finding the area. We can convert either $1\frac{1}{2}$ feet to 18 inches or 6 inches to $\frac{1}{2}$ foot.

Method 1

$$1. A = bh \quad b = 18 \text{ in.}, h = 6 \text{ in.}$$

$$2. A = 18 \times 6$$

$$3. A = 108$$

Answer: 108 sq. in.

Method 2

$$1. A = bh \quad b = 1\frac{1}{2} \text{ ft.}, h = \frac{1}{2} \text{ ft.}$$

$$2. A = 1\frac{1}{2} \times \frac{1}{2}$$

$$3. A = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$$

Answer: $\frac{3}{4}$ sq. ft.

NOTE. Unless otherwise stated, in area problems in which measurements are given in terms of a variable such as x , we will assume that all linear measurements were made with the same linear unit and all area measurements were made with the corresponding area unit.

2. A rectangle is inscribed in a circle whose radius is 5 inches. If the base of the rectangle is 8 inches, find the area of the rectangle.

Solution: Let the length of altitude $\overline{BC} = h$.

$$1. \text{ In rt. } \triangle ABC, (h)^2 + (8)^2 = (10)^2$$

$$2. \quad \quad \quad h^2 + 64 = 100$$

$$3. \quad \quad \quad h^2 = 36$$

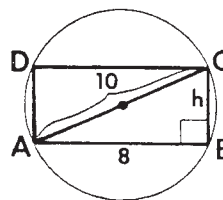
$$4. \quad \quad \quad h = 6$$

$$5. \text{ Area} = bh$$

$$6. \quad A = 8 \times 6$$

$$7. \quad A = 48$$

Answer: 48 sq. in.



EXERCISES

1. In each part, find the area of the rectangle whose dimensions are given.
 - a. $b = 10$ in., $h = 9$ in.
 - b. $b = 5$ ft., $h = 8.4$ ft.
 - c. $b = 12$ in., $h = 5\frac{1}{2}$ ft.
 - d. $b = 28$ in., $h = 7\frac{3}{4}$ in.
 - e. $b = 2$ ft., $h = 6$ in.
 - f. $b = 6\frac{1}{3}$ ft., $h = 1\frac{1}{2}$ ft.
2. Mr. Korman has a rectangular vegetable garden. He decides to increase the size by making the length twice that of the original garden with the width 3 times that of the original garden. How many times as large as the original garden will the area of the new garden be?
3. Find the base of a rectangle whose area is 48 sq. in. and whose altitude is 4 in.
4. The dimensions of a rectangular living room are 13 feet by 20 feet. How many square feet of carpet are needed to cover the whole floor?
5. How many tiles, each of which is 1 sq. ft., are needed to cover the floor of a pool which is 40 ft. wide and 80 ft. long?
6. Which rectangle has the greater area: one which is 12 ft. by 9 ft. or one which is 10 ft. by 11 ft.?
7. The perimeter of a rectangle is 30 ft. Find the area of the rectangle if one of its sides is (a) 4 ft. (b) 8 ft. (c) 36 in. (d) $2\frac{1}{2}$ ft.
8. A diagonal of a rectangle is 10 in. and the base is 6 in. Find the area of the rectangle.
9. A rectangle is inscribed in a circle whose diameter is 13 in. If one side of the rectangle is 5 in., find the area of the rectangle.
10. The base and the altitude of rectangle $PQRS$ are double those of rectangle $ABCD$. The area of $PQRS$, compared with the area of $ABCD$, is (1) the same (2) twice as great (3) four times as great (4) eight times as great.
11. If the base and the altitude of a rectangle are both tripled, the ratio of the area of the original rectangle to the area of the enlarged rectangle is (1) 1:3 (2) 1:6 (3) 1:9 (4) 1:18.
12. Patrick wants to plant small trees alongside his family's driveway. He wants the distance from the first tree to the last tree to be 200 feet and the trees to be 50 feet apart. How many trees does he need to buy?
13. Express, in terms of x , the areas of the rectangles which have the following dimensions:
 - a. $b = 3x$, $h = 4x$
 - b. $b = 7$, $h = 2x - 3$
 - c. $b = 3x$, $h = 3x + 1$
 - d. $b = x + 6$, $h = x - 2$
14. In a rectangle whose area is 35, the base is 5 and the altitude is $3x + 1$. (a) Find the value of x . (b) Find the altitude of the rectangle.
15. In a rectangle whose area is 300, the base and altitude are in the ratio 3:4. Find the dimensions of the rectangle.

16. The area of a rectangle is 56. If its base is represented by $x + 5$ and its height by $x - 5$, find its dimensions.
17. In a rectangle whose area is 15, the base is 1 less than twice the altitude. Find its dimensions.
18. A rectangle whose base is twice as large as its altitude is inscribed in a circle whose radius is 5. Find the area of the rectangle.
19. The perimeter of a rectangle is P and its altitude is H . (a) Express its base B in terms of P and H . (b) Express its area in terms of P and H . (c) State whether the following statement is *true* or *false*: All rectangles which have equal perimeters have equal areas.
20. The base of a rectangle is represented by x and the perimeter is 20. (a) Represent the altitude of the rectangle in terms of x . (b) Represent the area of the rectangle in terms of x .
21. Find the dimensions of a rectangle whose area is 20 sq. ft. and whose perimeter is 18 ft.
22. The base of a rectangle is 10 in. A diagonal of the rectangle makes an angle of 32° with the base of the rectangle. (a) Find the altitude of the rectangle to the *nearest tenth of an inch*. (b) Find the area of the rectangle to the *nearest square inch*.
23. Find the area of a rectangle whose diagonal is 50 if the angle which the diagonal makes with the base measures: [Answers may be left in radical form.] a. 30° b. 45° c. 60°
24. In rectangle $ABCD$, \overline{AB} is the base, and \overline{BD} is a diagonal whose length is 8 in. The perpendicular line drawn from C to \overline{BD} meets \overline{BD} at E . If $\overline{BE} \cong \overline{ED}$: (a) Find DC and CB . (b) Show that rectangle $ABCD$ is a square. (c) Find the area of square $ABCD$.
25. Rectangle $ABCD$ has the longer side \overline{AD} as base and diagonal \overline{AC} . The perpendicular line drawn from B to \overline{AC} meets \overline{AC} at E . If $BE = 6\sqrt{3}$, and EC exceeds AE by 12: (a) Find AE and EC . (b) Find the area of rectangle $ABCD$.

4. Finding the Area of a Square

Theorem 102. The area of a square of side s is equal to the square of s .

In Fig. 7-7, the area, A , of square $PQRS$, with side s , is given by the formula $A = s^2$.

Corollary T102-1. The area of a square of diagonal d is equal to one-half the square of d .

In Fig. 7-7, the area, A , of square $PQRS$, with diagonal d , is given by the formula $A = \frac{1}{2}d^2$.

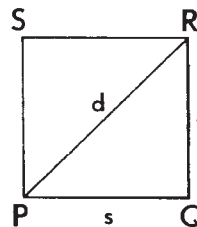


Fig. 7-7

Definition. Polygons having the same area are called *equivalent polygons*.

NOTE. The symbol that is used to indicate that two triangles are congruent (\cong) consists of two parts: $=$ to indicate that the triangles are equal in area; and \sim to indicate that the triangles are similar in shape.

MODEL PROBLEMS

1. Find the area of a square whose perimeter is 20 ft.

Solution:

1. $p = 4s$ $p = 20$.
2. $20 = 4s$
3. $s = 5$
4. $A = s^2 = (5)^2 = 25$

Answer: 25 sq. ft.

2. A rectangle the lengths of whose base and altitude are in the ratio 4:1 is equivalent to a square the length of whose side is 6 inches. Find the dimensions of the rectangle.

Solution: Let x = the altitude of the rectangle in inches.

Then $4x$ = the base of the rectangle in inches.

- | | |
|-----------------------------|---------------------------|
| 1. Area of rectangle = bh | 4. Area of square = s^2 |
| 2. $A = 4x \cdot x$ | 5. $A' = (6)^2$ |
| 3. $A = 4x^2$ | 6. $A' = 36$ |

Since the rectangle and the square are equivalent, their areas are equal.

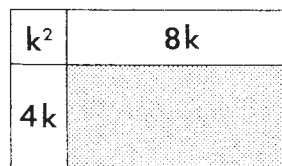
7. $A = A'$
8. $4x^2 = 36$
9. $x^2 = 9$
10. $x = 3, x = -3$ Reject the negative value.
11. $4x = 12$

Answer: Base of rectangle = 12 in., altitude = 3 in.

EXERCISES

1. Find the area of a square whose side is:
- a. 25 in. b. 13 ft. c. 9 yd. d. $\frac{2}{3}$ ft. e. 7.5 ft.

2. Express, in terms of x , the area of a square whose side is represented by:
 $a. x$ $b. 2x$ $c. x + 2$ $d. x - 3$ $e. 2x + 1$
3. Find the side of a square whose area is:
 $a. 144$ sq. in. $b. 49$ sq. ft. $c. .81$ sq. yd. $d. 2\frac{1}{4}$ sq. ft.
4. Find the area of a square whose perimeter is:
 $a. 80$ in. $b. 24$ in. $c. 32$ yd. $d. 18$ in. $e. 9$ ft.
5. A baseball diamond is a square 90 ft. on a side. Find the area enclosed by the baselines.
6. Find the area of a square whose diagonal is:
 $a. 8$ $b. 12$ $c. 5$ $d. \sqrt{2}$ $e. 3\sqrt{2}$ $f. 4\sqrt{2}$
7. How many square tiles 2 in. by 2 in. are needed to cover a bathroom floor which is 6 ft. by 8 ft.?
8. If the length of each side of a square is tripled, the ratio of the area of the original square to the area of the enlarged square is (1) 1:3 (2) 1:6 (3) 1:9 (4) 1:18.
9. If the ratio of the area of square $ABCD$ to the area of square $EFGH$ is 16:1, then the ratio of a side of square $ABCD$ to a side of square $EFGH$ is (1) 16:1 (2) 4:1 (3) 256:1 (4) 64:1.
10. Find the area of a square inscribed in a circle whose diameter is 8.
11. Find the area of a square circumscribed about a circle whose diameter is 20 in.
12. A square is equal in area to a rectangle whose base is 20 and whose altitude is 5. Find a side of the square.
13. Find a side of a square that is equivalent to a rectangle whose base is 4 inches and whose altitude is 3 feet.
14. A rectangle the length of whose base and altitude are in the ratio 5:1 is equivalent to a square whose perimeter is 40 in. Find the length of the base and the length of the altitude of the rectangle.
15. If s represents the length of a side of a square, b and h represent the lengths of the base and altitude of a rectangle, and the square and the rectangle are equivalent, express b in terms of s and h .
16. A square is circumscribed about a circle whose radius is R . Express the area of the square in terms of R .
17. A square is inscribed in a circle whose radius is R . Express the area of the square in terms of R .
18. In the figure, the large rectangle has been divided into a square and three smaller rectangles. If the areas of the square and two of the rectangles are k^2 , $4k$, and $8k$ respectively, what is the numerical value of the area of the shaded rectangle?



Ex. 18

5. Finding the Area of a Parallelogram

Any side of a parallelogram may be called its *base*. The *altitude* corresponding to this base is a segment which is perpendicular to the line containing this base and which is drawn from any point on the opposite side of the parallelogram.

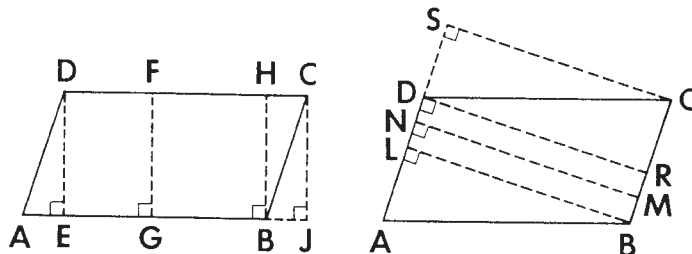


Fig. 7-8

In parallelogram $ABCD$, if \overline{AB} is considered as the base, shown at the left in Fig. 7-8, then any one of the parallel, congruent segments, \overline{DE} , \overline{FG} , \overline{HB} , or \overline{CJ} , can be considered a corresponding altitude; if \overline{AD} is considered as the base, shown at the right in Fig. 7-8, then any one of the parallel, congruent segments, \overline{BL} , \overline{MN} , \overline{RD} , or \overline{CS} , can be considered a corresponding altitude.

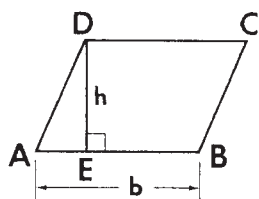


Fig. 7-9

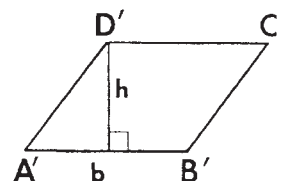
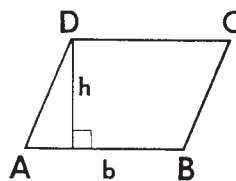


Fig. 7-10

Theorem 103. The area of a parallelogram is equal to the product of the length of any base and the length of any corresponding altitude.

[The proof for this theorem appears on pages 766–767.]

In Fig. 7-9, the area, A , of parallelogram $ABCD$, the length of whose base is represented by b and the length of whose altitude drawn to base b is represented by h , is given by the formula $A = bh$.

Corollary T103-1. Parallelograms which have congruent bases and congruent corresponding altitudes are equal in area.

In Fig. 7-10, parallelograms $ABCD$ and $A'B'C'D'$, which have congruent bases and congruent corresponding altitudes, are equal in area.

Corollary T103-2. The area of a parallelogram is equal to the product of the lengths of two consecutive sides and the sine of their included angle.

In Fig. 7-11, the area, A , of parallelogram $CDEF$, whose two consecutive sides \overline{CF} and \overline{CD} include angle C , is given by the formula $A = ab \sin C$.

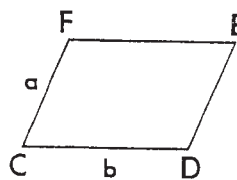


Fig. 7-11

MODEL PROBLEMS

- Find the area of a parallelogram whose base is 8 ft. and whose altitude is 18 in.

Solution: Convert 18 in. to $1\frac{1}{2}$ ft.

- $A = bh$ $b = 8$ ft., $h = 1\frac{1}{2}$ ft.

- $A = 8 \times 1\frac{1}{2} = 12$

Answer: 12 sq. ft.

- The lengths of two consecutive sides of a parallelogram are 10 inches and 15 inches, and these sides include an angle of 63° .
 - Find to the *nearest tenth of an inch* the length of the altitude drawn to the longer side of the parallelogram.
 - Find to the *nearest square inch* the area of the parallelogram.

Solution: Let h = the length of the altitude.

- $\sin 63^\circ = \frac{h}{10}$

- $0.8910 = \frac{h}{10}$

- $h = 10(0.8910)$

- $h = 8.910$

- $h = 8.9$ to the nearest tenth of an inch

Answer: 8.9 in.

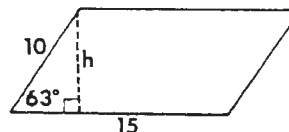
- $A = bh$

- $A = 15(8.9)$

- $A = 133.5$

- $A = 134$ to the nearest square inch

Answer: 134 sq. in.



- A parallelogram whose base is represented by $x + 4$ and whose altitude is represented by $x - 1$ is equivalent to a square whose side is 6. Find the base and altitude of the parallelogram.

[The solution is given on the next page.]

Solution:

1. Area of parallelogram $= bh$.
2. $A = (x + 4)(x - 1)$.
3. Area of square $= s^2$.
4. $A' = 6^2 = 36$.

Since the parallelogram and the square are equivalent, their areas are equal.

5. $A = A'$
6. $(x + 4)(x - 1) = 36$
7. $x^2 + 3x - 4 = 36$
8. $x^2 + 3x - 40 = 0$
9. $(x - 5)(x + 8) = 0$
10. $x - 5 = 0 \quad | \quad x + 8 = 0$
11. $x = 5 \quad | \quad x = -8$ Reject the negative value.
12. $x + 4 = 5 + 4 = 9$
13. $x - 1 = 5 - 1 = 4$

Answer: Base of parallelogram $= 9$, altitude $= 4$.

EXERCISES

1. If a and b represent the sides of a parallelogram and h represents the altitude on side b , the formula for the area K of the parallelogram is $K = \underline{\hspace{2cm}}$.
2. Find the area of the parallelogram whose base and altitude have the following measurements:

a. $b = 8$ in., $h = 12$ in.	b. $b = 7$ ft., $h = 5.3$ ft.
c. $b = 18$ in., $h = 6\frac{1}{2}$ in.	d. $b = 32$ in., $h = 5\frac{1}{4}$ in.
e. $b = 4$ ft., $h = 9$ in.	f. $b = 8\frac{1}{2}$ ft., $h = 2\frac{1}{3}$ ft.
3. Two parallelograms have the same base. If their altitudes are in the ratio 4:5, what is the ratio of their areas?
4. Express, in terms of x , the areas of the parallelograms whose bases and altitudes are represented by:

a. $b = 3x$, $h = 5$	b. $b = 3x$, $h = 2x$
c. $b = x - 3$, $h = 4$	d. $b = 10$, $h = 3x + 2$
e. $b = x - 2$, $h = x$	f. $b = x + 6$, $h = x - 3$

5. In a parallelogram whose area is 60, a side is represented by $4x - 4$ and the altitude drawn to that side is 5. Find the value of x and the length of the side represented by $4x - 4$.
6. In a parallelogram, the ratio of a side to the altitude drawn to that side is 2:1. If the area of the parallelogram is 50, find the altitude.
7. In a parallelogram whose area is 36, a side is 5 less than the altitude drawn to that side. Find the altitude.
8. The area of a parallelogram is 30 and a base is 10. Find the altitude drawn to the base.
9. Find the area of a parallelogram if one base angle contains 30° and the measures of the sides including this angle are:
a. 4 and 8 *b.* 12 and 20 *c.* 10 and 5 *d.* 9 and 8
10. Find the area of a parallelogram if one base angle contains 60° and the measures of the sides including this angle are: [Answers may be left in radical form.]
a. 8 and 14 *b.* 18 and 20 *c.* 6 and 9 *d.* 13 and 16
11. Find the area of a parallelogram if one base angle contains 45° and the measures of the sides including this angle are: [Answers may be left in radical form.]
a. 10 and 6 *b.* 12 and 16 *c.* 8 and 11 *d.* 15 and 4
12. Find the area of a parallelogram two of whose consecutive sides are 8 and 20 and include an angle of 150° .
13. Find the area of a parallelogram two of whose adjacent sides are 10 and 6 and include an angle of 120° . [Answer may be left in radical form.]
14. Represent, in terms of x , the area of a parallelogram if one base angle contains 30° and the measures of the sides including this angle are represented by:
a. 4 and $2x$ *b.* 6 and $5x$ *c.* 3 and x *d.* $4x$ and $2x$
15. Represent, using x , the area of a parallelogram two of whose adjacent sides are represented by 6 and $4x$ and which include an angle of:
a. 60° *b.* 45° *c.* 150° *d.* 120° *e.* 135°
16. A parallelogram is equal in area to a square whose side is 8. If a side of the parallelogram and the altitude drawn to that side are in the ratio 4:1, find the side.
17. In parallelogram $ABCD$, \overline{AE} is perpendicular to \overline{BC} , and \overline{AF} is perpendicular to \overline{CD} . If $BC = 10$, $CD = 8$, and $AE = 4$, find AF .
18. A parallelogram is equal in area to a rectangle. The altitude of the rectangle is 6 and its diagonal is 10. The ratio of the altitude of the parallelogram to the altitude of the rectangle is 2:1. (a) Find the base of the parallelogram. (b) Select the correct completion for the following statement: The perimeter of the rectangle is (1) greater than the perimeter of the parallelogram (2) equal to the perimeter of the parallelogram (3) smaller than the perimeter of the parallelogram.

19. In parallelogram $ABCD$, angle A measures 45° , altitude \overline{DE} on base \overline{AB} measures 8, and diagonal \overline{DB} measures 17. (a) Find AE . (b) Find EB . (c) Find the area of the parallelogram.
20. In each part of this question, find the altitude drawn to the larger side of the parallelogram to the *nearest tenth*. Use this result to find the area of the parallelogram to the *nearest integer*.
Two adjacent sides of a parallelogram are 8 and 10 and they include an angle of:
a. 28° b. 35° c. 80° d. 120° e. 150° f. 170°
21. The area of a parallelogram is 72 sq. in. The longer side measures 12 in. and one of the angles of the parallelogram measures 48° . Find to the *nearest tenth of an inch* the shorter side of the parallelogram.
22. The base of a parallelogram is 16 and one base angle measures 36° . If the area of the parallelogram is 80, find to the *nearest tenth* the side adjacent to the base.
23. In parallelogram $ABCD$, \overline{DE} is an altitude to base \overline{AB} , $m\angle A = 67^\circ$, $AD = 13$, and $BD = 20$. (a) Find DE to the *nearest integer*. (b) Find AE to the *nearest integer*. (c) Using the values found in answer to parts a and b , find BE and the area of $ABCD$.

6. Finding the Area of a Triangle

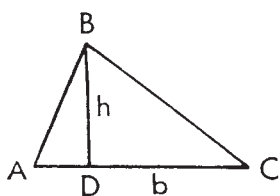


Fig. 7-12

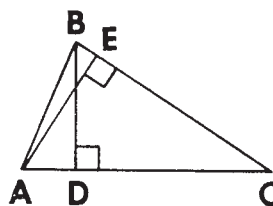


Fig. 7-13

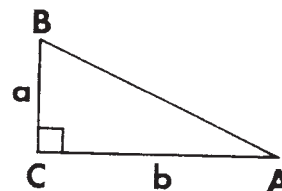


Fig. 7-14

Theorem 104. The area of a triangle is equal to one-half the product of the length of a side and the length of the altitude drawn to that side.

[The proof for this theorem appears on pages 767–768.]

NOTE. Any side of a triangle may be called a *base* of the triangle. For example, in $\triangle ABC$ (Fig. 7-13), \overline{AC} may be called a base of $\triangle ABC$. \overline{BD} would then be called the corresponding altitude drawn to the base. Also, \overline{CB} may be called a base of the triangle. Then \overline{AE} would be called the corresponding altitude drawn to the base.

In Fig. 7-12, the area, A , of triangle ABC , the length of whose base \overline{AC} is represented by b and the length of whose altitude \overline{BD} drawn to base \overline{AC} is represented by h , is given by the formula $A = \frac{1}{2}bh$.

Corollary T104-1. In any triangle, the product of the length of any side and the length of the altitude drawn to that side is equal to the product of the length of any other side and the length of the altitude drawn to that side.

In triangle ABC (Fig. 7-13), $AC \times BD = BC \times AE$.

Corollary T104-2. The area of a right triangle is equal to one-half the product of the lengths of its two legs.

In right triangle ABC (Fig. 7-14), whose legs are \overline{BC} and \overline{AC} , the area, A , is given by the formula $A = \frac{1}{2} \times BC \times AC$, or $A = \frac{1}{2}ab$.

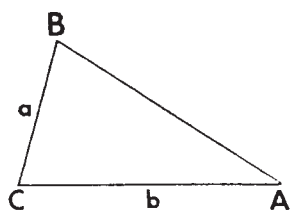


Fig. 7-15

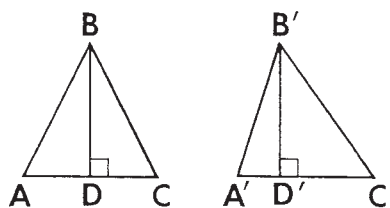


Fig. 7-16

Corollary T104-3. The area of a triangle is equal to one-half the product of the lengths of two consecutive sides and the sine of their included angle.

In Fig. 7-15, the area, A , of triangle ABC , whose two consecutive sides \overline{AC} and \overline{CB} include angle C , is given by the formula $A = \frac{1}{2}ab \sin C$.

Corollary T104-4. Triangles which have congruent bases and congruent altitudes which are drawn to those bases are equal in area.

In Fig. 7-16, triangles ABC and $A'B'C'$, which have congruent bases, $\overline{AC} \cong \overline{A'C'}$, and congruent altitudes, $\overline{BD} \cong \overline{B'D'}$, are equal in area (equivalent), denoted by $\triangle ABC = \triangle A'B'C'$.

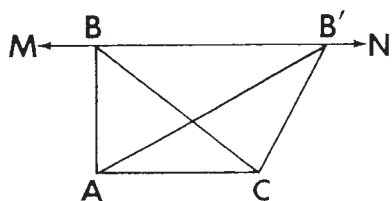


Fig. 7-17

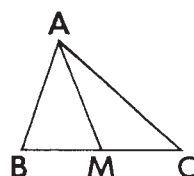


Fig. 7-18

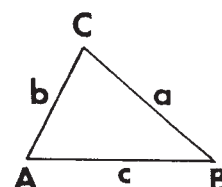


Fig. 7-19

Corollary T104-5. Triangles which have a common base and whose vertices lie on a line parallel to the base are equal in area.

In Fig. 7-17, triangles ABC and $AB'C$, which have \overline{AC} as a common base and whose vertices B and B' lie on line \overleftrightarrow{MN} , which is parallel to \overline{AC} , are equal in area.

Corollary T104-6. A median drawn to a side of a triangle divides the triangle into two triangles which are equal in area.

In triangle ABC (Fig. 7-18), if \overline{AM} is a median drawn to \overline{BC} , then triangle BAM is equal in area to triangle CAM , denoted by $\triangle BAM = \triangle CAM$.

Theorem 105. The area of a triangle the length of whose three sides are represented by a , b , and c is given by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

[s represents the semi-perimeter: $s = \frac{1}{2}(a + b + c)$.]

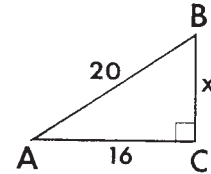
In $\triangle ABC$ (Fig. 7-19), $A = \sqrt{s(s-a)(s-b)(s-c)}$. This formula is known as *Heron's formula* or *Hero's formula* for the area of a triangle.

MODEL PROBLEMS

1. In right triangle ABC , $m\angle C = 90$, $AB = 20$ in., and $AC = 16$ in. Find the area of triangle ABC .

Solution: Consider \overline{AC} as the base of triangle ABC .

Since $m\angle C = 90$, \overline{BC} is the altitude to \overline{AC} in triangle ABC . Let $x =$ the length of \overline{BC} .



1. $(x)^2 + (16)^2 = (20)^2$
2. $x^2 + 256 = 400$
3. $x^2 = 144$
4. $x = 12$
5. Area of $\triangle ABC = \frac{1}{2}bh$
6. Area of $\triangle ABC = \frac{1}{2}(16)(12) = 96$

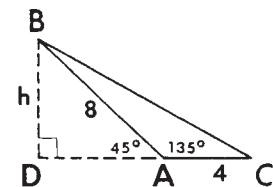
Answer: Area of triangle $ABC = 96$ sq. in.

NOTE. The area of right triangle ABC was found by using the corollary "The area of a right triangle is equal to one-half the product of the lengths of its two legs."

2. Find the area of triangle ABC to the nearest square inch if $AB = 8$ in., $AC = 4$ in., and $m\angle A = 135$.

Solution: If \overline{CA} is considered as the base of triangle ABC , the altitude drawn to \overline{CA} extended is BD . Let $h =$ length of \overline{BD} .

1. In right triangle BAD , $m\angle BAD = 180 - 135 = 45$.
2. $h = \frac{1}{2}(\text{hypotenuse})\sqrt{2}$
3. $h = \frac{1}{2}(8)\sqrt{2} = 4\sqrt{2}$
4. Area of $\triangle ABC = \frac{1}{2}bh = \frac{1}{2}(4)(4\sqrt{2})$



5. Area of $\triangle ABC = 8\sqrt{2} = 8(1.4) = 11.2$ [Use $\sqrt{2} = 1.4$.]

Answer: Area of triangle $ABC = 11$ sq. in.

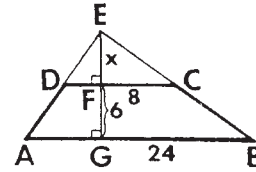
3. In trapezoid $ABCD$, the larger base \overline{AB} measures 24 in., the smaller base \overline{DC} measures 8 in., and altitude \overline{FG} measures 6 in. The nonparallel sides \overline{AD} and \overline{BC} are extended to meet at E .
- In triangle DEC , find EF , the measure of the altitude from E to \overline{DC} .
 - Find the area of triangle DEC .

Solution:

1. In trapezoid $ABCD$, $\overline{DC} \parallel \overline{AB}$. $DC = 8$. Let $EF = x$.
 $AB = 24$. Then $EG = x + 6$.
 $FG = 6$.

3. $\frac{\text{length of altitude } \overline{EF}}{\text{length of altitude } \overline{EG}} = \frac{\text{length of base } \overline{DC}}{\text{length of base } \overline{AB}}$, or

$$\frac{x}{x+6} = \frac{8}{24}$$



- $24x = 8x + 48$
- $16x = 48$
- $x = 3$

Answer: $EF = 3$ in.

1. Area of $\triangle DEC = \frac{1}{2}bh = \frac{1}{2}DC \times EF$
2. Area of $\triangle DEC = \frac{1}{2} \times 8 \times 3 = 12$

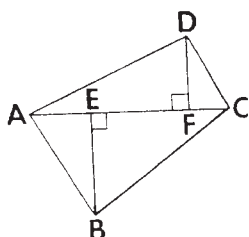
Answer: Area of $\triangle DEC = 12$ sq. in.

EXERCISES

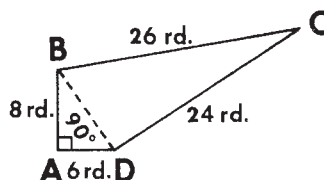
- Find the area of a triangle whose base and altitude have the following measures:
 - $b = 12$ in., $h = 8$ in.
 - $b = 14$ in., $h = 9$ in.
 - $b = 21$ in., $h = 1\frac{1}{2}$ ft.
 - $b = 7$ ft., $h = 11$ ft.
 - $b = 3$ ft., $h = 8$ in.
 - $b = 2\frac{1}{2}$ ft., $h = 3\frac{1}{4}$ ft.
- Express, in terms of x , the areas of the triangles whose bases and altitudes are represented by:
 - $b = 2x$, $h = 6$
 - $b = 5x$, $h = 4x$
 - $b = 12$, $h = 5x - 2$
 - $b = 2x + 4$, $h = 3x$
- The area of a triangle is 60. If one side of the triangle measures 24, find the length of the altitude drawn to that side.

4. The area of a triangle is 40. If a side of the triangle is represented by $2x + 2$ and the altitude drawn to that side measures 8, find the value of x .
5. The area of a triangle is 12. If the ratio of the length of a side of the triangle to the length of the altitude drawn to that side is 2:3, find the length of the side of the triangle.
6. The area of a triangle is 20. If a side of the triangle measures 3 less than the altitude drawn to that side, find the altitude.
7. Find the area of a right triangle whose legs measure:
a. 6 and 8 b. 12 and 9
8. Find the area of an isosceles right triangle each of whose legs measures 6.
9. If the length of each leg of an isosceles right triangle is represented by L , express the area of the triangle in terms of L .
10. Find the area of an isosceles right triangle whose hypotenuse measures $4\sqrt{2}$.
11. The sides of a triangle have lengths of 3, 4, and 5. Find the area of the triangle.
12. A side of a triangle measures 12 and the altitude to this side measures 4. A second side of the triangle measures 16. Find the altitude drawn to this side.
13. The lengths of the sides of a right triangle measure 5, 12, and 13. Find the length of the altitude drawn to the hypotenuse of the triangle.
14. The bases of two equivalent triangles measure 20 in. and 40 in. Find the ratio of the lengths of the altitudes which are drawn to the given bases of the triangles.
15. In $\triangle ABC$, \overline{CM} is the median drawn to side \overline{AB} . If the area of $\triangle ACM$ is 15 sq. in., find the area of $\triangle BCM$.
16. A triangle is equal in area to a square whose side measures 12. If the base of the triangle has a length of 36, find the length of the altitude drawn to this side of the triangle.
17. Find the area of a triangle if one of its angles contains 30° and the sides including this angle measure:
a. 6 and 10 b. 30 and 40 c. 20 and 9 d. 11 and 8 e. 5 and 7
18. Find the area of a triangle if one of its angles contains 60° and the sides including this angle measure: [Answers may be left in radical form.]
a. 12 and 6 b. 20 and 14 c. 10 and 9 d. 13 and 8 e. 3 and 5
19. Find the area of a triangle if one of its angles contains 45° and the sides including this angle measure: [Answers may be left in radical form.]
a. 8 and 6 b. 14 and 12 c. 4 and 10 d. 11 and 10 e. 5 and 9

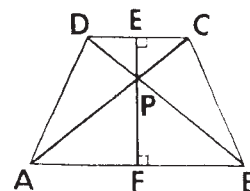
20. Find the area of a triangle if one of its angles contains 150° and the lengths of the sides including this angle are:
a. 4 and 6 b. 8 and 5 c. 7 and 9
21. Find the area of a triangle if one of its angles contains 135° and the sides including this angle measure 6 and 8. [Answers may be left in radical form.]
22. Represent, in terms of x , the area of a triangle if one of its angles contains 30° and the lengths of the sides including this angle are represented by $4x$ and $2x$.
23. Represent, in terms of x , the area of a triangle if two of its sides are represented by 8 and $2x$ and the angle included between these sides contains:
a. 60° b. 45° c. 150° d. 120° e. 135°
24. Triangle ABC is equal in area to a square whose side measures 4. If $m\angle BAC = 30$ and AB is 4 times AC , find AC and AB .
25. Find the area of an isosceles triangle whose base measures 8 and each of whose congruent sides measures 5.
26. \overline{AC} , a diameter of a circle, is 8 inches long and forms an angle of 30° with chord \overline{AB} . Find the area of triangle ABC .
27. In triangle ABC , \overline{AC} and \overline{BC} are each 13 and \overline{AB} is 10. (a) Find the length of the altitude upon \overline{AB} . (b) Find the area of triangle ABC . (c) Find the length of the altitude from B upon \overline{AC} .
28. In circle O , chord \overline{AB} is 16 inches long. C is the midpoint of minor arc \widehat{AB} , and diameter \overline{CE} intersects chord \overline{AB} in D . \overline{CD} is 4 inches long. (a) Find the length of \overline{DE} . (b) Find the length of a diameter of the circle. (c) Draw radii \overline{OA} and \overline{OB} . Find the area of triangle AOB .
29. The bases of a trapezoid measure 7 and 10 and the altitude measures 6. (a) Find the length of the altitude of the triangle formed by the shorter base and the nonparallel sides extended. (b) Find the area of the triangle described in a.
30. In isosceles trapezoid $ABCD$, the length of base \overline{AB} is 60 feet, the length of base \overline{CD} is 28 feet, and the length of leg \overline{BC} is 20 feet. Legs \overline{AD} and \overline{BC} are extended to intersect in E . Find the area of triangle DEC .
31. In trapezoid $ABCD$, the length of base \overline{AB} is 12 and the length of base \overline{CD} is 4. If diagonal \overline{AC} is drawn, find the ratio of the area of triangle ABC to the area of triangle ACD .
32. Given square $ABCD$ with E , F , G , and H the midpoints of \overline{AB} , \overline{BC} , \overline{CD} , and \overline{DA} respectively.
a. Prove triangle $EFG \cong$ triangle EHG .
b. If $AB = 6$, find (1) the length of \overline{EF} and (2) the area of $\triangle EFG$.



Ex. 33



Ex. 34



Ex. 36

33. To find the area of quadrilateral $ABCD$, diagonal \overline{AC} is drawn, and \overline{BE} and \overline{DF} are drawn perpendicular to \overline{AC} . If $AC = 24$ ft., $BE = 12$ ft., and $DF = 9$ ft., find the area of quadrilateral $ABCD$.
34. A plot of land has the form of a quadrilateral whose sides are 8 rods, 26 rods, 24 rods, and 6 rods, as indicated on the accompanying figure. Angle BAD contains 90° . (a) Find the length of diagonal \overline{BD} . (b) Show that angle BDC is a right angle. (c) Find to the nearest tenth of an acre the area of the field. [1 acre = 160 sq. rd.]
35. In trapezoid $ABCD$, \overline{AB} is the longer base. Diagonals \overline{AC} and \overline{BD} intersect in E .
- Prove: $\triangle AEB \sim \triangle CED$.
 - If the bases of the trapezoid measure 5 inches and 15 inches, find the ratio of corresponding altitudes of $\triangle CED$ and $\triangle AEB$.
 - If the altitude of the trapezoid measures 8 inches, find the number of square inches in the area of $\triangle AEB$.
36. In trapezoid $ABCD$, $AB = 20$, $DC = 10$, and altitude \overline{EF} measures 12. (a) Using x to represent PE , write an equation which can be used to find PE . (b) Solve this equation for x . (c) Find the area of triangle DPC and triangle BPA .
37. In trapezoid $ABCD$, \overline{AB} is the lower base. The bisector of angle A intersects \overline{DC} at P . (a) Prove that triangle ADP is isosceles. (b) If $AD = 8$ and the altitude of the trapezoid measures 6, find the area of triangle ADP .
38. The lengths of the legs of a right triangle are in the ratio 3:4. If the area of the triangle is 54, find the lengths of the three sides of the triangle.
39. In triangle RST , the lengths of sides \overline{RS} , \overline{ST} , and \overline{RT} are represented by $2x + 9$, $5x - 3$, and $4x$ respectively. The perimeter of triangle RST is 50. (a) Find the lengths of the sides of triangle RST . (b) Find the area of triangle RST .
40. In each part of this exercise, find the length of the altitude drawn to the larger side to the nearest tenth. Use this result to find the area of the triangle to the nearest integer.

Two adjacent sides of a triangle measure 6 and 10 and include an angle of:

- a. 25° b. 37° c. 75° d. 120° e. 150° f. 175°

41. Find to the *nearest square inch* the area of an isosceles triangle whose vertex angle contains 120° and each of whose legs measures 12 inches.
42. In an isosceles triangle, the vertex angle contains 120° and the length of each of the congruent legs is represented by L . Represent the area of the triangle in terms of L .
43. a. Given acute $\triangle ABC$ with the sides opposite angles A , B , and C represented by a , b , and c , respectively. Starting with the formula $K = \frac{1}{2}bh$ for the area of $\triangle ABC$, show that $K = \frac{1}{2}ab \sin C$ is also a formula for the area of $\triangle ABC$.
 b. Using the formula $K = \frac{1}{2}ab \sin C$, find the number of degrees contained in angle C if angle C is acute, $a = 40$, $b = 10$, and $K = 100$.
44. In right triangle ABC , the length of hypotenuse \overline{AB} is 13.7 and angle B contains 38° . (a) Find AC and BC to the *nearest tenth*. (b) Find the area of triangle ABC to the *nearest integer*.
45. In an isosceles triangle, the base is 24 inches and each base angle contains 50° . (a) Find to the *nearest tenth of an inch* the altitude drawn upon the base. (b) Find the area of the triangle to the *nearest square inch*.
46. The altitude of a triangle is 12 inches, and it divides the vertex angle into two angles containing 20° and 45° . (a) Find the lengths of the segments of the base. (b) Find to the *nearest square inch* the area of the triangle.

7. Finding the Area of an Equilateral Triangle

All the theorems which involve the area of the general triangle also apply to the equilateral triangle. In addition, the following theorem can be used in finding the area of an equilateral triangle:

Theorem 106. The area of an equilateral triangle, the length of each of whose sides is represented by s , is given by the formula $A = \frac{s^2}{4} \sqrt{3}$.

MODEL PROBLEMS

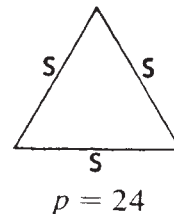
1. Find the area of an equilateral triangle whose perimeter is 24.

Solution:

1. Perimeter is 24. Therefore, the length of each side is $24 \div 3 = 8$.

$$2. A = \frac{s^2}{4} \sqrt{3} = \frac{(8)^2}{4} \sqrt{3} = \frac{64}{4} \sqrt{3} = 16\sqrt{3}.$$

Answer: Area = $16\sqrt{3}$.



2. Find the length of a side of an equilateral triangle whose area is $4\sqrt{3}$.

Solution:

$$1. \quad A = \frac{s^2}{4} \sqrt{3}$$

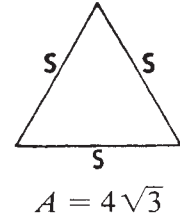
$$2. \quad 4\sqrt{3} = \frac{s^2}{4} \sqrt{3}$$

$$3. \quad 4 = \frac{s^2}{4}$$

$$4. \quad 16 = s^2$$

$$5. \quad s = 4$$

Answer: Side = 4.

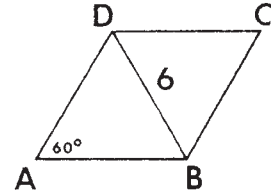


3. Find the area of a rhombus one of whose angles contains 60° and whose shorter diagonal is 6.

Solution:

1. Since $ABCD$ is a rhombus, $\overline{BA} \cong \overline{AD}$.

2. In $\triangle DAB$, $m\angle A = 60$ and $\overline{BA} \cong \overline{AD}$. Therefore, $m\angle ABD = m\angle BDA = 60$, and $\triangle DAB$ is an equilateral triangle.



$$3. \quad \text{Area of } \triangle DAB = \frac{s^2}{4} \sqrt{3} = \frac{(6)^2}{4} \sqrt{3} = \frac{36}{4} \sqrt{3} = 9\sqrt{3}$$

4. Since \overline{BD} divides rhombus $ABCD$ into two equivalent triangles, area of rhombus $ABCD = 2 \times \text{area of } \triangle DAB = 2(9\sqrt{3}) = 18\sqrt{3}$.

Answer: Area = $18\sqrt{3}$.

EXERCISES

1. Express the area of an equilateral triangle in terms of the length of its side b .
2. Find the area of an equilateral triangle the length of whose side is: [Answers may be left in radical form.]

a. 2	b. 4	c. 6	d. 10	e. 12	f. 20
g. 3	h. 1	i. 5	j. 9	k. 15	l. 25

3. Find the area of an equilateral triangle whose perimeter is: [Answers may be left in radical form.]
 a. 6 b. 12 c. 18 d. 30 e. 15 f. 21
4. Find the length of a side of an equilateral triangle whose area is:
 a. $9\sqrt{3}$ b. $25\sqrt{3}$ c. $16\sqrt{3}$ d. $100\sqrt{3}$ e. $\sqrt{3}$
 f. $\frac{9}{4}\sqrt{3}$ g. $\frac{25}{4}\sqrt{3}$ h. $\frac{49}{4}\sqrt{3}$ i. $\frac{81}{4}\sqrt{3}$ j. $\frac{\sqrt{3}}{4}$
5. In triangle ABC , the lengths of sides \overline{AB} , \overline{BC} , and \overline{CA} are represented by $3x - 3$, $x + 7$, and $2x + 2$ respectively. The perimeter of triangle ABC is 36. (a) Find the lengths of the sides of triangle ABC . (b) Find the area of triangle ABC . [Answer may be left in radical form.]
6. The area of an equilateral triangle is $36\sqrt{3}$. Find the length of the radius of the circle inscribed in this triangle. [Answer may be left in radical form.]
7. Find the area of a rhombus one of whose angles contains 60° and whose shorter diagonal measures:
 a. 4 b. 8 c. 12 d. 3 e. 5
8. Find the area of a rhombus one of whose angles contains 60° and whose longer diagonal measures 12.
9. The area of a regular hexagon is $96\sqrt{3}$. Find the length of a side of an equilateral triangle whose perimeter is equal to the perimeter of the hexagon.

8. Finding the Area of a Trapezoid

Theorem 107. The area of a trapezoid is equal to one-half the product of the length of its altitude and the sum of its bases.

[The proof for this theorem appears on pages 768–769.]

In Fig. 7-20, the area of trapezoid $ABCD$, the lengths of whose bases \overline{AB} and \overline{DC} are represented by b and b' and the length of whose altitude \overline{DE} is represented by h , is given by the formula

$$A = \frac{1}{2}h(b + b')$$

Corollary T107-1. The area of a trapezoid is equal to the product of the length of its altitude and the length of its median.

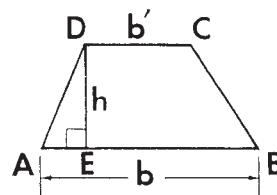


Fig. 7-20

In Fig. 7-21, the area of trapezoid $ABCD$, the length of whose altitude \overline{DE} is represented by h and the length of whose median \overline{FG} is represented by m , is given by the formula $A = h \times m$.

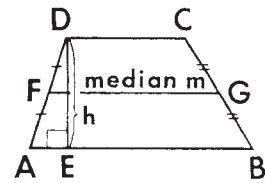


Fig. 7-21

MODEL PROBLEMS

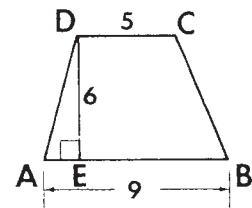
1. Find the area of trapezoid $ABCD$ if the length of $\overline{AB} = 9$ in., the length of $\overline{DC} = 5$ in., and the length of altitude $\overline{DE} = 6$ in.

Solution:

$$1. A = \frac{1}{2}h(b + b') \quad b = 9, b' = 5, h = 6.$$

$$2. A = \frac{1}{2}(6)(9 + 5) = 3(14) = 42$$

Answer: Area = 42 sq. in.



2. The bases of a trapezoid are 12 in. and 20 in. If the area of the trapezoid is 128 sq. in., find the length of its altitude.

Solution:

$$1. \text{ Let } h = \text{the length of altitude } \overline{DE}. \quad b = 20, b' = 12, A = 128.$$

$$2. A = \frac{1}{2}h(b + b')$$

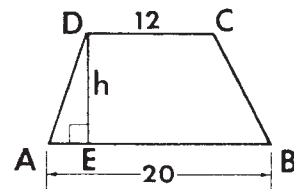
$$3. 128 = \frac{1}{2}h(20 + 12)$$

$$4. 128 = \frac{1}{2}h(32)$$

$$5. 128 = 16h$$

$$6. 8 = h$$

Answer: Altitude = 8 in.



3. In isosceles trapezoid $ABCD$, \overline{CF} and \overline{DE} are altitudes and $m\angle B = 60^\circ$. CD exceeds BC by 5. If the perimeter of $ABCD$ is 110, find, in radical form, the area of the trapezoid.

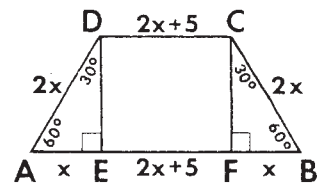
Solution:

1. In right triangle CBF , since $m\angle FBC = 60^\circ$, $m\angle BCF = 30^\circ$.

2. In 30° - 60° right triangle CBF , $BC = 2FB$.

3. Let $FB = x$. Then $BC = 2x$.

4. Therefore, $CD = 2x + 5$.



5. Since $DCFE$ is a rectangle, $EF = CD = 2x + 5$.
6. Since $\triangle DAE \cong \triangle CBF$, $AD = BC = 2x$ and $AE = FB = x$.
7. $AE + EF + FB + BC + CD + DA = \text{perimeter of } ABCD$
8. $x + 2x + 5 + x + 2x + 2x + 5 + 2x = 110$
9. $10x + 10 = 110$
10. $10x = 100$
11. $x = 10$
12. $CD = 2x + 5 = 2(10) + 5 = 25$, $AB = 4x + 5 = 4(10) + 5 = 45$.
13. In 30° - 60° right triangle CBF , $CF = \frac{1}{2}BC\sqrt{3} = \frac{1}{2}(20)\sqrt{3} = 10\sqrt{3}$.
14. Area of trapezoid $ABCD = \frac{1}{2}h(b + b')$
15. Area of trapezoid $ABCD = \frac{1}{2}(10\sqrt{3})(45 + 25)$
 $= \frac{1}{2}(10\sqrt{3})(70) = \frac{1}{2}(700\sqrt{3}) = 350\sqrt{3}$

Answer: $350\sqrt{3}$.

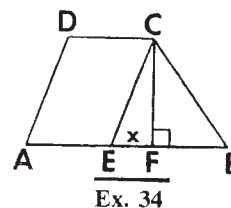
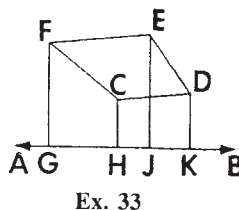
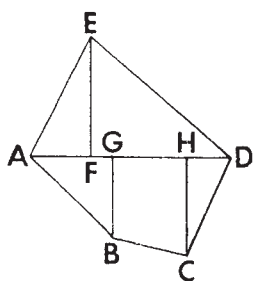
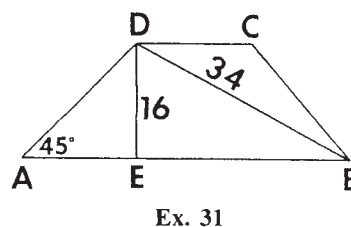
EXERCISES

1. The formula for the area A of a trapezoid in terms of its altitude h and its bases b and b' is $A = \underline{\hspace{2cm}}$.
2. The altitude of a trapezoid is 6 and its bases are 8 and 12. Find the area of the trapezoid.
3. The bases of a trapezoid are 2.4 inches and 5.6 inches and its altitude is 7.0 inches. Find the number of square inches in the area of the trapezoid.
4. The bases of a trapezoid are 8 and 12 and its area is 140. Find its altitude.
5. The area of a trapezoid is 36 and the sum of its bases is 18. Find the length of the altitude.
6. Find the area of a trapezoid whose altitude measures 8 and whose median measures 14.
7. Find the area of a trapezoid whose altitude measures 13 and whose median measures 30.
8. The area of a trapezoid is 72. Its altitude measures 8. Find the lengths of its bases if the larger base is twice the smaller base.
9. The area of a trapezoid is 90 square inches. The altitude measures 6 inches and one base measures 18 inches. Find the number of inches in the length of the other base.

10. The area of a trapezoid is 54 square inches. Its altitude measures 6 in. If the larger base exceeds the smaller base by 2 in., find the lengths of both bases of the trapezoid.
11. If the bases of an isosceles trapezoid are 20 in. and 28 in. respectively and each leg measures 5 in., find the number of square feet in the area of the trapezoid.
12. The congruent sides of an isosceles trapezoid each measure 5 and its altitude measures 4. If the area of the trapezoid is 48, find the lengths of the bases.
13. In trapezoid $ABCD$, the lengths of the bases \overline{AB} and \overline{DC} are 30 and 20, $m\angle A = 30$, and $AD = 8$. Find the area of the trapezoid.
14. In trapezoid $ABCD$, the lengths of the bases \overline{AB} and \overline{DC} are 15 and 11, $m\angle A = 60$, and $AD = 4$. Find the area of the trapezoid. [Answer may be left in radical form.]
15. In trapezoid $ABCD$, the lengths of the bases \overline{AB} and \overline{DC} are 17 and 14, $m\angle A = 45$, and $AD = 6$. Find the area of the trapezoid. [Answer may be left in radical form.]
16. In isosceles trapezoid $ABCD$, the bases are \overline{AB} and \overline{DC} . AD is 1 more than twice DC , and AB is 6 more than 4 times DC . (a) If DC is represented by x , represent AD and AB in terms of x . (b) If the perimeter of the trapezoid is 62, find the value of x . (c) Find the length of the altitude of the trapezoid. (d) Find the area of the trapezoid.
17. Find the area of an isosceles trapezoid whose bases are 10 and 26 and whose base angles each contain 60° . [Answer may be left in radical form.]
18. In an isosceles trapezoid, each base angle contains 45° and the bases measure 8 and 24. Find the area of the trapezoid.
19. The measure of the longer base of an isosceles trapezoid exceeds the measure of the shorter base by 6, and each leg measures 5. (a) If the measure of the shorter base is represented by x , express the measure of the longer base in terms of x . (b) Find the measure of the altitude of the trapezoid. (c) If the area of this trapezoid equals 28, find the measure of each base.
20. (a) The longer base of an isosceles trapezoid is 21, its altitude is 6, and one of its angles contains 45° . Find the area of the trapezoid. (b) The length of base \overline{AD} of parallelogram $ABCD$ is represented by x . If angle A contains 30° and if the length of side \overline{AB} is 10, express the area of the parallelogram in terms of x . (c) If the trapezoid and the parallelogram are equal in area, find x .
21. A trapezoid is inscribed in a circle whose radius is 13 inches in length. The bases are 12 inches and 5 inches from the center of the circle and

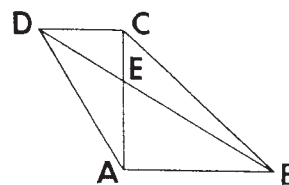
- on opposite sides of the center. (a) Find the lengths of the bases of the trapezoid. (b) Find the area of the trapezoid.
22. In trapezoid $ABCD$, the length of base \overline{AB} is 6 inches and the length of base \overline{DC} is 24 inches. Sides \overline{DA} and \overline{CB} are extended to meet at point G . The altitude of the trapezoid is 6 inches longer than the altitude to side \overline{AB} of triangle GAB . (a) Using x to represent the length of the altitude from G in triangle GAB , represent the length of the corresponding altitude in triangle GDC . (b) Find the length of the altitude of the trapezoid. (c) Find the area of the trapezoid.
23. The base of a triangle measures 30 inches and the altitude drawn to this base measures 15 inches. Find the area of the trapezoid formed by a line parallel to the base and 9 inches from the opposite vertex.
24. Each side of a triangle is 8 inches. A line is drawn parallel to one side of the triangle and forming a trapezoid one of whose nonparallel sides is 6 inches. (a) Find the length of the altitude of the trapezoid. (b) Find to the nearest square inch the area of the trapezoid.
25. In trapezoid $ABCD$, the length of base \overline{BC} is to the length of base \overline{AD} as 2 is to 3. Legs \overline{AB} and \overline{DC} are extended to meet at E , and the altitude \overline{EF} of triangle AED intersects \overline{BC} at G . The area of triangle AED is 270 and $EF = 30$. (a) Find AD . (b) Find BC and EG . (c) Find the area of $ABCD$.
26. The diameter \overline{AB} of circle O is a base of the inscribed trapezoid $ABCD$. $m\angle A = 60$ and the length of radius \overline{OD} is 8. (a) Find the length of the altitude of the trapezoid. (b) Find the area of the trapezoid.
27. The area of an equilateral triangle is equal to that of a trapezoid whose bases are 4 and 14 and whose altitude measures $4\sqrt{3}$. Find the length of a side of the triangle.
28. Find the side of an equilateral triangle which is equal in area to an isosceles trapezoid whose bases measure 6 and 10 and each of whose base angles contains 60° .
29. A rectangle and a trapezoid are equal in area. One side of the rectangle is 20 inches and its diagonal is 25 inches. The altitude of the trapezoid is 12 inches and one base is 10 inches longer than the other. (a) Find the area of the rectangle. (b) If x represents the length of the shorter base of the trapezoid, express the area of the trapezoid in terms of x . (c) Find x .
30. $ABCD$ is a trapezoid with bases \overline{AB} and \overline{DC} . Diagonals \overline{CA} and \overline{DB} intersect in O . AB is 20, DC is 4, and the area of the trapezoid is 72. (a) Find the altitude of the trapezoid. (b) Find the length of the perpendicular drawn from O to \overline{AB} .

31. In the figure, the bases of isosceles trapezoid $ABCD$ are \overline{AB} and \overline{DC} , with \overline{AB} the longer base. The length of altitude \overline{DE} is 16, the length of diagonal \overline{DB} is 34, and $m\angle A = 45^\circ$.
- Find the number of units in the length of \overline{AE} , \overline{EB} , \overline{AB} and \overline{DC} .
 - Find the number of square units in the area of trapezoid $ABCD$.



32. In the figure, $\overline{EF} \perp \overline{AD}$, $\overline{BG} \perp \overline{AD}$, and $\overline{CH} \perp \overline{AD}$. $AF = 7$, $FG = 3$, $GH = 9$, $HD = 5$, $FE = 14$, $BG = 10$, $HC = 12$. Find the area of $ABCDE$.
33. In the figure, $\overline{FG} \perp \overline{AB}$, $\overline{CH} \perp \overline{AB}$, $\overline{EJ} \perp \overline{AB}$, and $\overline{DK} \perp \overline{AB}$. $GH = 14$, $HJ = 6$, $JK = 8$, $FG = 20$, $CH = 8$, $EJ = 22$, and $DK = 10$. Find the area of $CDEF$.
34. The lengths of the bases \overline{AB} and \overline{CD} of trapezoid $ABCD$ are 24 and 10, and the lengths of the legs \overline{AD} and \overline{BC} are 13 and 15 respectively. \overline{CE} is drawn parallel to \overline{DA} , and \overline{CF} is perpendicular to \overline{AB} . Let EF be represented by x .
- Express FB in terms of x .
 - Using triangles EFC and BFC , write *two* expressions for $(CF)^2$ in terms of x .
 - Find the value of x .
 - Find the area of the trapezoid.
35. The lengths of the bases of an isosceles trapezoid are 8 and 28. One base angle contains 53° . (a) Find to the *nearest tenth* the length of the altitude of the trapezoid. (b) Find to the *nearest integer* the area of the trapezoid.
36. In trapezoid $ABCD$, the length of base $\overline{AB} = 23$, the length of base $\overline{DC} = 17$, the length of leg $\overline{AD} = 10$, and $m\angle A = 40^\circ$. (a) Find to the *nearest tenth* the length of the altitude of the trapezoid. (b) Find to the *nearest integer* the area of the trapezoid.

37. In the figure, $ABCD$ is a trapezoid in which $\overline{AB} \parallel \overline{DC}$. Diagonal $\overline{CA} \perp$ base \overline{AB} . $AB = 28.0$ in., $DC = 12.0$ in., and the measure of the angle included between diagonal \overline{BD} and base \overline{AB} is 24° . (a) Find the length of \overline{AC} to the nearest tenth of an inch. (b) Find the area of trapezoid $ABCD$ to the nearest square inch, using the answer obtained in part a.



Ex. 37

38. In trapezoid $ABCD$, angle A contains 55° and bases \overline{AB} and \overline{DC} are perpendicular to leg \overline{BC} . The length of base \overline{DC} is 18 and the length of leg \overline{AD} is 14. (a) Find the length of the altitude of the trapezoid to the nearest tenth. (b) Using the answer obtained in part a, find the area of the trapezoid to the nearest integer.

9. Finding the Area of a Rhombus

All the theorems which involve the area of the general parallelogram also apply to the rhombus. In addition, the following theorem can be used in finding the area of a rhombus.

Theorem 108. The area of a rhombus is equal to one-half the product of the lengths of its diagonals.

In Fig. 7-22, the area of rhombus $ABCD$, the lengths of whose diagonals \overline{AC} and \overline{BD} are represented by d_1 and d_2 , is given by the formula $A = \frac{1}{2}d_1d_2$.

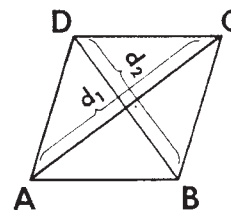


Fig. 7-22

MODEL PROBLEMS

1. Find the area of a rhombus each of whose sides is 10 in. and one of whose diagonals is 16 in.

Solution:

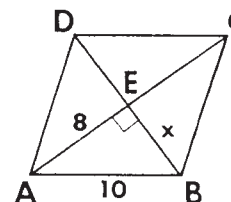
1. Since the diagonals of rhombus $ABCD$ bisect each other at right angles,

$$\overline{AE} \perp \overline{BD}, \text{ and } AE = EC = \frac{1}{2}AC = \frac{1}{2}(16) = 8$$

2. In right $\triangle AEB$, let $BE = x$. Thus,

$$x^2 + 8^2 = (10)^2, x^2 + 64 = 100, x^2 = 36, x = 6. \quad AB = 10 \text{ in.}$$

[The solution continues on the next page.]



$$3. BD = 2BE = 2(6) = 12.$$

$$4. \text{Area of rhombus } ABCD = \frac{1}{2}d_1d_2 = \frac{1}{2}(16)(12) \quad AC = 16 \text{ in.} \\ = 8(12) = 96$$

Answer: Area of rhombus $ABCD = 96$ sq. in.

2. The area of a rhombus is 90 and one diagonal is 10. Find the other diagonal.

Solution:

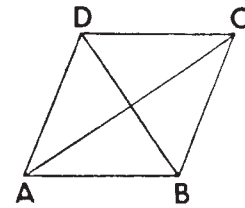
$$1. A = \frac{1}{2}d_1d_2 \quad A = 90.$$

$$2. 90 = \frac{1}{2}(10)(x) \quad \text{Let } BD = d_1 = 10.$$

$$3. 90 = 5x \quad \text{Let } AC = d_2 = x.$$

$$4. 18 = x$$

Answer: Diagonal = 18.

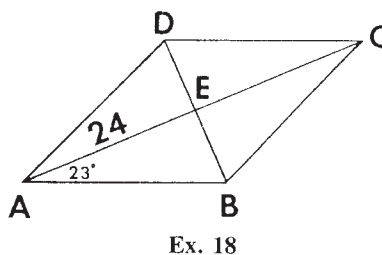


EXERCISES

- The area, A , of a rhombus in terms of the lengths of its diagonals d and d' is _____.
- Find the area of a rhombus whose diagonals measure:
 - 6 and 8
 - 4 and 5
 - 25 and 15
 - 6.4 and 3.5
 - 4 and $4\sqrt{3}$
- If the diagonals of a rhombus measure 8 and 12, the area of the rhombus is
 - 24
 - 48
 - 96
 - 10.
- The area of a rhombus is 54 and one of its diagonals measures 12. Find the length of the other diagonal.
- Find the area of a rhombus whose sides measure 5 and one of whose diagonals measures 8.
- Find a side of a square equal in area to a rhombus whose diagonals measure 9 and 8.
- Find the length of the altitude in a rhombus whose area is 320 and whose base is 20.
- The lengths of the diagonals of a rhombus are represented by n and $n + 3$. Express the area of the rhombus in terms of n .
- One diagonal of a rhombus is twice as long as the other. If the area of the rhombus is 100 square inches, find the number of inches in the length of the shorter diagonal.
- The diagonals of a rhombus measure 15 and 20. The rhombus is equal in

area to a trapezoid whose altitude measures 10. If one base of the trapezoid is twice the second base, find the bases of the trapezoid.

11. The perimeter of a rhombus is 40 and one of its diagonals measures 12. Find the area of the rhombus.
12. The perimeter of a rhombus is 52 and one of its diagonals measures 10. Find the area of the rhombus.
13. One angle of a rhombus contains 60° and a side measures 4. Find the area of the rhombus.
14. The shorter diagonal of a rhombus is equal in length to one of its sides. The length of a side of the rhombus is 6 inches. (a) Find the length of each diagonal of the rhombus. (b) Find the area of the rhombus.
15. If one angle of a rhombus contains 120° and its shorter diagonal measures 8, find its area.
16. The diagonals of a rhombus measure 10 and 24. (a) Find the length of a side of the rhombus. (b) Find the area of the rhombus. (c) Find the length of the altitude of the rhombus.
17. If one side of a rhombus is 25 and the length of the longer diagonal is 40, find: (a) the length of the shorter diagonal (b) the area of the rhombus (c) the length of the altitude.
18. In the figure, $ABCD$ is a rhombus. Diagonal \overline{AC} makes an angle of 23° with side \overline{AB} , and $AE = 24$.
 - a. Find the length of \overline{EB} to the nearest integer.
 - b. Using the result found in answer to a, find the area of the rhombus.
 - c. Find the length of \overline{AB} to the nearest integer.
 - d. Find to the nearest integer the length of the altitude of the rhombus from D to side \overline{AB} .
19. In rhombus $ABCD$, $m\angle A = 38$ and the length of diagonal $\overline{DB} = 14.0$ inches. (a) Find to the nearest tenth of an inch the length of diagonal \overline{AC} . (b) Find to the nearest square inch the area of the rhombus. (c) Find the length of side \overline{AD} to the nearest inch.
20. The lengths of the diagonals of a rhombus are in the ratio 3:4. The area of the rhombus is 96. Find the length of each diagonal and the perimeter of the rhombus.
21. The area of a rhombus is 64 and the length of one diagonal is twice the other. Find the length of the shorter diagonal.
22. The length of the larger diagonal of a rhombus exceeds the length of the smaller diagonal by 2. The area of the rhombus is 40. Find the lengths of the diagonals of the rhombus.



Ex. 18

10. Proving Triangles and Polygons Equal in Area

We have learned that two triangles (polygons) are *equivalent* if they are equal in area.

When two triangles (polygons) are congruent (\cong), they are both similar (\sim) and equal in area ($=$).

KEEP IN MIND

The following corollaries, which we have previously studied, can be helpful in proving triangles, also polygons, equivalent:

1. A diagonal of a parallelogram divides the parallelogram into two congruent triangles.
2. Two triangles are equivalent if they have congruent bases and congruent altitudes.
3. Two triangles are equivalent if they have a common base and their vertices lie on a line parallel to the base.
4. A median drawn to a side of a triangle divides the triangle into two equivalent triangles.

MODEL PROBLEMS

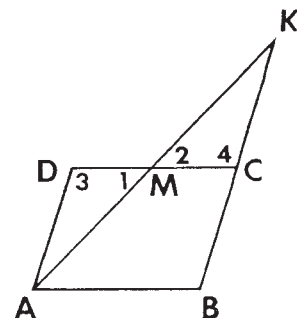
1. In parallelogram $ABCD$, M is the midpoint of side \overline{DC} . Line segment \overline{AM} extended intersects \overline{BC} extended at K .

- a. Prove that triangle ADM is congruent to triangle KCM .
- b. Prove that triangle AKB is equal in area to parallelogram $ABCD$.

Given: $ABCD$ is a \square .
 \overleftrightarrow{KA} and \overleftrightarrow{KB} are straight lines.
 $\overline{DM} \cong \overline{CM}$.

To prove: (a) $\triangle ADM \cong \triangle KCM$.
 (b) Area of $\triangle AKB$ = area of $\square ABCD$.

- a. Plan: Prove that $\triangle ADM \cong \triangle KCM$ by showing that a.s.a. \cong a.s.a.



<i>Proof: Statements</i>	<i>Reasons</i>
1. $ABCD$ is a \square .	1. Given.
2. \overleftrightarrow{KA} and \overleftrightarrow{KB} are straight lines.	2. Given.
3. $\angle 1 \cong \angle 2$. (a. \cong a.)	3. If two angles are vertical angles, they are congruent.
4. $\overline{AD} \parallel \overline{BC}$.	4. The opposite sides of a parallelogram are parallel.
5. $\angle 3 \cong \angle 4$. (a. \cong a.)	5. If two parallel lines are cut by a transversal, the alternate interior angles are congruent.
6. $\overline{DM} \cong \overline{CM}$. (s. \cong s.)	6. Given.
7. $\triangle ADM \cong \triangle KCM$.	7. a.s.a. \cong a.s.a.

b. Plan: To prove that the area of $\triangle AKB$ = the area of $\square ABCD$, show that the sum of the areas of quadrilateral $ABCM$ and $\triangle KCM$ is equal to the sum of the areas of quadrilateral $ABCM$ and $\triangle ADM$.

<i>Proof: Statements</i>	<i>Reasons</i>
1. $\triangle KCM \cong \triangle ADM$.	1. Proved in part <i>a</i> .
2. Area of $\triangle KCM$ = area of $\triangle ADM$.	2. If two triangles are congruent, they are equal in area.
3. Area of $ABCM$ = area of $ABCM$.	3. Reflexive property of equality.
4. Area of $\triangle KCM$ + area of $ABCM$ = area of $\triangle ADM$ + area of $ABCM$.	4. Addition property of equality.
5. Area of $\triangle KCM$ + area of $ABCM$ = area of $\triangle AKB$. Area of $\triangle ADM$ + area of $ABCM$ = area of $\square ABCD$.	5. Area-addition postulate.
6. Area of $\triangle AKB$ = area of $\square ABCD$.	6. Substitution postulate.

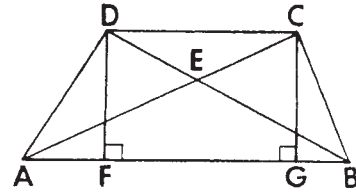
2. \overline{AB} and \overline{DC} are the bases of trapezoid $ABCD$. Diagonals \overline{AC} and \overline{BD} intersect in E . Prove that triangle ADE is equal in area to triangle CEB .

[Model Problem 2 continues on the next page.]

Given: Trapezoid $ABCD$ with bases \overline{AB} and \overline{DC} .

To prove: Area of $\triangle ADE$ = area of $\triangle CEB$.

Plan: Prove that the area of $\triangle ADB$ = the area of $\triangle ACB$. Show that the difference between the areas of $\triangle ADB$ and $\triangle AEB$ (area of $\triangle ADB$ - area of $\triangle AEB$) is equal to the difference between the areas of $\triangle ACB$ and $\triangle AEB$ (area of $\triangle ACB$ - area of $\triangle AEB$).



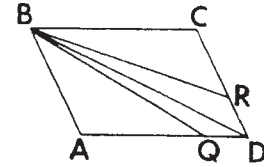
<i>Proof:</i>	<i>Statements</i>	<i>Reasons</i>
1.	Draw $\overline{DF} \perp \overline{AB}$ and $\overline{CG} \perp \overline{AB}$.	1. An altitude may be drawn to a side of a triangle.
2.	\overline{AB} and \overline{DC} are bases of trapezoid $ABCD$.	2. Given.
3.	$\overline{DC} \parallel \overline{AB}$.	3. The bases of a trapezoid are parallel.
4.	$DF = CG$, or $\overline{DF} \cong \overline{CG}$.	4. Parallel lines are everywhere equidistant.
5.	$\overline{AB} \cong \overline{AB}$.	5. Reflexive property of congruence.
6.	Area of $\triangle ADB$ = area of $\triangle ACB$.	6. Two triangles which have congruent bases and congruent altitudes are equal in area.
7.	Area of $\triangle AEB$ = area of $\triangle AEB$.	7. Reflexive property of equality.
8.	Area of $\triangle ADB$ - area of $\triangle AEB$ = area of $\triangle ACB$ - area of $\triangle AEB$, or	8. Subtraction postulate of equality.
9.	Area of $\triangle ADE$ = area of $\triangle CEB$.	9. Substitution postulate.

EXERCISES

1. In parallelogram $ABCD$, diagonals \overline{AC} and \overline{BD} are drawn. Prove that $\triangle ADB$ is equal in area to $\triangle ACB$.

2. In trapezoid $ABCD$, \overline{AB} is the larger base and \overline{DC} is the smaller base. Diagonals \overline{AC} and \overline{BD} are drawn. Prove $\triangle ACD$ is equal in area to $\triangle BCD$.
3. In $\triangle RST$, median \overline{TN} is drawn to side \overline{RS} and median \overline{RK} is drawn to side \overline{ST} . Prove that $\triangle RTN$ is equal in area to $\triangle RKS$.
4. In $\triangle ABC$, median \overline{CD} is drawn to side \overline{AB} . P is any point on median \overline{CD} between C and D . \overline{PA} and \overline{PB} are drawn. (a) Prove: area of $\triangle APD$ = area of $\triangle BPD$. (b) Prove: area of $\triangle APC$ = area of $\triangle BPC$.
5. In $\triangle ABC$, D is the midpoint of \overline{AB} and E is the midpoint of \overline{AC} . Prove: area of $\triangle CDB$ = area of $\triangle CEB$.
6. In $\triangle ABC$, D is the midpoint of \overline{AC} , E is the midpoint of \overline{BC} , and F is the midpoint of \overline{AB} . Prove that $\triangle DEF$ is equivalent to $\triangle DAF$.
7. The diagonals of a parallelogram intersect in a point forming four triangles. Prove that the four triangles are equal in area.
8. In parallelogram $ABCD$, E is a point on diagonal \overline{AC} . Through E , a line is drawn parallel to \overline{AB} , intersecting \overline{AD} in F and \overline{BC} in K . Through E , another line is drawn parallel to \overline{AD} , intersecting \overline{AB} in G and \overline{DC} in H . (a) Prove that triangle HEC is congruent to triangle KCE . (b) Prove that the area of quadrilateral $AEHD$ is equal to the area of quadrilateral $AEKB$.
9. In parallelogram $ABCD$, perpendiculars drawn to diagonal \overline{AC} from B and D meet \overline{AC} at points E and K respectively. (a) Prove that $\overline{BE} \cong \overline{DK}$. (b) A point H is taken on \overline{AC} , and \overline{BH} and \overline{DH} are drawn. Prove that triangle ABH is equal in area to triangle ADH .
10. $ABCD$ is a parallelogram. E is the midpoint of diagonal \overline{BD} . Through E , a line is drawn intersecting \overline{BC} in F and \overline{AD} in G . Prove: (a) Triangle DEG is congruent to triangle BEF . (b) The area of quadrilateral $ABEG$ is equal to the area of quadrilateral $CDEF$.
11. In quadrilateral $ABCD$, M is the midpoint of diagonal \overline{AC} . \overline{MB} and \overline{MD} are drawn. Prove that $ADMB$ is equal in area to $CDMB$.
12. If diagonal \overline{AC} of quadrilateral $ABCD$ bisects diagonal \overline{BD} , prove that $\triangle ABC$ is equal in area to $\triangle ADC$.
13. In parallelogram $ABCD$, P is the point of intersection of diagonals \overline{AC} and \overline{BD} . Through P , a line is drawn which intersects \overline{AB} in S and \overline{CD} in R . Prove quadrilateral $ASRD$ is equal in area to quadrilateral $BSRC$.
14. In parallelogram $ABCD$, $\overline{DE} \perp \overline{AB}$ and $\overline{CF} \perp \overline{AB}$ extended through B . Prove that $ABCD$ is equal in area to $DEFC$.
15. In $\triangle RST$, M is any point on \overline{RT} and N is the midpoint of \overline{ST} . \overline{MN} is drawn and extended to P so that $\overline{MN} \cong \overline{NP}$. \overline{PS} is drawn. Prove that $\triangle RST$ is equal in area to quadrilateral $MRSP$.
16. In parallelogram $ABCD$, P , any point on \overline{CD} , is joined to A and B . Prove that the area of $\triangle ABP$ is one-half the area of parallelogram $ABCD$.

17. From any point P in the base \overline{AC} of triangle ABC , line segments are drawn to R and S , the midpoints of \overline{AB} and \overline{BC} respectively. Perpendiculars from R and S are drawn to \overline{AC} , terminating in \overline{AC} . (a) Prove that these perpendiculars are congruent. (b) Prove that the area of triangle ARP plus the area of triangle CSP equals one-half the area of triangle ABC .
18. In parallelogram $ABCD$, E is the midpoint of \overline{AD} and F is the midpoint of \overline{DC} . If \overline{BE} and \overline{BF} are drawn, prove that $\triangle BAE$ is equal in area to $\triangle BCF$.
19. Given parallelogram $ABCD$ with diagonal \overline{BD} . Q is a point on \overline{AD} and R is a point on \overline{CD} such that $AQ:QD = CR:RD = 2:1$.
- Prove that the area of triangle ABQ is $\frac{2}{3}$ of the area of triangle ABD .
 - Prove that line segments \overline{BQ} and \overline{BR} divide the parallelogram into three parts that are equal in area.
20. \overline{CD} is an altitude of triangle ABC and E is a point on \overline{CD} such that $DE = \frac{1}{3}CD$. Through E , a line parallel to \overline{AB} intersects \overline{AC} in K and \overline{BC} in L . P , the midpoint of \overline{KL} , is joined to A , B , and C . Prove that:
- area of $\triangle CKP$ = area of $\triangle CPL$
 - area of $\triangle CAP$ = area of $\triangle CBP$
 - area of $\triangle ABP = \frac{1}{3}$ of the area of $\triangle ABC$
 - area of $\triangle PAC$ = area of $\triangle PAB$ = area of $\triangle PBC$.



Ex. 19

11. Comparing Areas of Similar Triangles

Theorem 109. The ratio of the areas of two similar triangles is equal to the ratio of the squares of the lengths of any two corresponding sides.

In Fig. 7-23, if $\triangle ABC \sim \triangle A'B'C'$ and s and s' are the lengths of a pair of corresponding sides in these triangles, then

$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle A'B'C'} = \frac{(s)^2}{(s')^2}, \quad \text{or} \quad \frac{A}{A'} = \frac{(s)^2}{(s')^2}$$

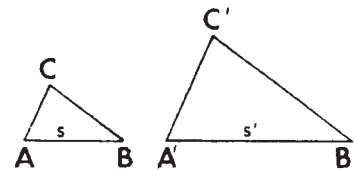


Fig. 7-23

Corollary T109-1. The ratio of the areas of two similar triangles is equal to the ratio of the squares of the lengths of any two corresponding line segments.

If $\triangle ABC \sim \triangle A'B'C'$ and l and l' are the lengths of any pair of corresponding line segments in these triangles, then

$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle A'B'C'} = \frac{(l)^2}{(l')^2}, \quad \text{or} \quad \frac{A}{A'} = \frac{(l)^2}{(l')^2}$$

Corollary T109-2. The ratio of the areas of two similar triangles is equal to the square of their ratio of similitude.

If $\triangle ABC \sim \triangle A'B'C'$ and $\frac{s}{s'}$ is the ratio of similitude in the two triangles, then

$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle A'B'C'} = \left(\frac{s}{s'}\right)^2, \quad \text{or} \quad \frac{A}{A'} = \left(\frac{s}{s'}\right)^2$$

MODEL PROBLEMS

1. Triangle $ABC \sim$ triangle $A'B'C'$. If $BC = 4$ and $B'C' = 12$, find the ratio of the areas of the triangles.

Solution:

$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle A'B'C'} = \frac{(BC)^2}{(B'C')^2} = \frac{(4)^2}{(12)^2} = \frac{16}{144} = \frac{1}{9}$$

Answer: Ratio of the areas of the triangles is 1:9.

2. The areas of two similar triangles are in the ratio of 4:1. The length of a side of the smaller triangle is 5. Find the length of the corresponding side in the larger triangle.

Solution:

Method 1

1. $\frac{A}{A'} = \frac{(s)^2}{(s')^2} \quad \frac{A}{A'} = \frac{4}{1}$
2. $\frac{4}{1} = \frac{s^2}{25} \quad s' = 5$
3. $s^2 = 100$
4. $s = 10$

Method 2

1. If the ratio of the areas is 4:1, the ratio of the lengths of any pair of corresponding sides is 2:1.
2. Since the length of a side in the smaller triangle = 5, the length of the corresponding side in the larger triangle = $2 \times 5 = 10$.

Answer: 10

EXERCISES

1. Find the ratio of the areas of two similar triangles in which the lengths of two corresponding sides are:
- a. $s = 1, s' = 5$ b. $s = 10, s' = 15$ c. $s = 9, s' = 3$

2. Find the ratio of the areas of two similar triangles in which the ratio of the lengths of a pair of corresponding lines is:
a. 4:1 *b.* 9:1 *c.* 7:1 *d.* 4:9 *e.* 3:5
3. The lengths of a pair of corresponding altitudes of two similar triangles are 4 inches and 2 inches. The area of the larger triangle is _____ times the area of the smaller.
4. In two similar triangles, the ratio of similitude is 2:5. Find the ratio of the areas of the two triangles.
5. The ratio of the lengths of a pair of corresponding sides of two similar triangles is 3:1. (*a*) The length of a side in the larger triangle is how many times as large as the length of the corresponding side of the smaller triangle? (*b*) The length of a side of the smaller triangle is what fractional part of the length of the corresponding side in the larger triangle? (*c*) The area of the larger triangle is how many times the area of the smaller triangle? (*d*) The area of the smaller triangle is what fractional part of the area of the larger triangle?
6. Find the ratio of the lengths of a pair of corresponding sides in two similar triangles if the ratio of their areas is:
a. 1:4 *b.* 1:25 *c.* 9:1 *d.* 4:9 *e.* 25:4
7. The ratio of the areas of two similar triangles is 25:16. (*a*) The area of the larger triangle is how many times the area of the smaller triangle? (*b*) The area of the smaller triangle is what fractional part of the area of the larger triangle? (*c*) The length of a side of the larger triangle is how many times the length of the corresponding side in the smaller triangle? (*d*) The length of a side of the smaller triangle is what fractional part of the length of the corresponding side in the larger triangle?
8. The areas of two similar triangles are in the ratio of 4:9. The length of one side of the smaller triangle is 4. Find the length of the corresponding side of the other triangle.
9. The areas of two similar triangles are in the ratio 9:1. The length of a side of the larger triangle is 12. What is the length of the corresponding side of the smaller triangle?
10. Two triangles are similar, and the area of one is four times the area of the other. The length of one side of the smaller triangle is 12. Find the length of the corresponding side of the larger triangle.
11. Two similar triangles have areas of 16 and 36. The length of a side of the smaller triangle is 8. Find the length of the corresponding side of the larger triangle.
12. The lengths of two corresponding medians of two similar triangles are 10 and 15. If the area of the larger triangle is 81, find the area of the smaller triangle.
13. Two equilateral triangles have sides that measure 8 in. and 12 in. respectively. Find the ratio of the areas of the two triangles.

14. The ratio of the areas of two equilateral triangles is 81 to 25. Find the ratio of the perimeters of the triangles.
15. In trapezoid $ABCD$, the length of base \overline{AB} is 20 in. and the length of base \overline{DC} is 15 in. If the nonparallel sides \overline{BC} and \overline{AD} are extended to meet in E , find the ratio of the area of triangle DEC to the area of triangle AEB .
16. In a circle, chords \overline{AB} and \overline{CD} intersect at E . Chords \overline{AD} and \overline{BC} are drawn. If $DE = 6$ and $BE = 2$, find the ratio of the area of $\triangle AED$ to the area of $\triangle BEC$.
17. *Prove:* A line segment which joins the midpoints of two sides of a triangle cuts off a triangle which is equal in area to one-fourth of the area of the given triangle.
18. Two triangles are similar. The area of the larger triangle exceeds the area of the smaller triangle by 40 square inches. (a) If A represents the area of the smaller triangle, represent the area of the larger triangle in terms of A . (b) If the ratio of the lengths of a pair of corresponding sides in the triangles is 3:1, write an equation that can be used to find A . (c) Find A .
19. The difference between the areas of two similar triangles is 90 square inches. (a) If the ratio of the area of the smaller triangle to the area of the larger triangle is 1:4, find the area of each triangle. (b) If the length of one side of the smaller triangle is 12 inches, find the length of the corresponding side of the larger triangle.
20. In the figure, \overline{FG} is parallel to \overline{BC} . The altitude \overline{AD} of triangle ABC is 6 and BC is 24. The ratio of the area of triangle AFG to the area of triangle ABC is 4:9.
- Find the length of \overline{FG} .
 - Find the length of the altitude of trapezoid $BCGF$.
21. $ABCD$ is an isosceles trapezoid with \overline{AB} the larger base and \overline{DC} the smaller base. $DC = 8$ and the length of each of the congruent sides, \overline{AD} and \overline{BC} , is equal to 5. Sides \overline{AD} and \overline{BC} are extended, intersecting in E and forming isosceles triangles AEB and DEC . The ratio of the area of triangle DEC to the area of triangle AEB is 1:4. (a) Find the length of \overline{DE} . (b) Find the length of the altitude drawn from E to \overline{AB} in triangle AEB . (c) Find the area of triangle AEB .



Ex. 20

12. Comparing Areas of Similar Polygons

Theorem 110. The ratio of the areas of two similar polygons is equal to the ratio of the squares of the lengths of any two corresponding sides.

In Fig. 7-24, if polygon $TUVWX \sim$ polygon $T'U'V'W'X'$, then

$$\frac{\text{area of polygon } TUVWX}{\text{area of polygon } T'U'V'W'X'} = \frac{(s)^2}{(s')^2},$$

or $\frac{A}{A'} = \frac{(s)^2}{(s')^2}$

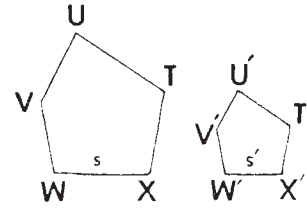


Fig. 7-24

Corollary T110-1. The ratio of the areas of two similar polygons is equal to the ratio of the squares of the lengths of any two corresponding line segments.

In Fig. 7-24, if polygon $TUVWX \sim$ polygon $T'U'V'W'X'$ and l and l' are the lengths of any pair of corresponding line segments in these polygons, then

$$\frac{\text{area of polygon } TUVWX}{\text{area of polygon } T'U'V'W'X'} = \frac{(l)^2}{(l')^2}, \quad \text{or} \quad \frac{A}{A'} = \frac{(l)^2}{(l')^2}$$

Corollary T110-2. The ratio of the areas of two similar polygons is equal to the square of their ratio of similitude.

In Fig. 7-24, if polygon $TUVWX \sim$ polygon $T'U'V'W'X'$ and $\frac{s}{s'}$ is the ratio of similitude in the two polygons, then

$$\frac{\text{area of polygon } TUVWX}{\text{area of polygon } T'U'V'W'X'} = \left(\frac{s}{s'}\right)^2, \quad \text{or} \quad \frac{A}{A'} = \left(\frac{s}{s'}\right)^2$$

MODEL PROBLEMS

1. Polygon $WXYZ \sim$ polygon $W'X'Y'Z'$. If $WX = 3$ and $W'X' = 12$, find the ratio of the areas of the two polygons.

Solution:

$$\frac{\text{area of polygon } WXYZ}{\text{area of polygon } W'X'Y'Z'} = \frac{(WX)^2}{(W'X')^2} = \frac{(3)^2}{(12)^2} = \frac{9}{144} = \frac{1}{16}$$

Answer: Ratio of the areas of the two polygons is 1:16.

2. The lengths of two corresponding sides of two similar polygons are 4 and 6. If the area of the smaller polygon is 20, find the area of the larger polygon.

Solution:

<i>Method 1</i>	<i>Method 2</i>
1. $\frac{A}{A'} = \frac{(s)^2}{(s')^2}$ $s = 4$, $s' = 6$, $A = 20$.	1. The ratio of similitude is $\frac{s}{s'} = \frac{4}{6} = \frac{2}{3}$
2. $\frac{20}{x} = \frac{(4)^2}{(6)^2}$ Let $x = A'$.	2. $\frac{A}{A'} = \left(\frac{s}{s'}\right)^2$
3. $\frac{20}{x} = \frac{16}{36}$	3. $\frac{20}{x} = \left(\frac{2}{3}\right)^2$
4. $16x = 720$	4. $\frac{20}{x} = \frac{4}{9}$
5. $x = 45$	5. $4x = 180$
	6. $x = 45$

Answer: Area of the larger polygon is 45.

EXERCISES

- Find the ratio of the areas of two similar polygons in which the lengths of two corresponding sides are:
 a. $s = 4$, $s' = 8$ b. $s = 8$, $s' = 12$ c. $s = 10$, $s' = 5$
- The length of a side of a polygon is three times as large as the length of the corresponding side of a similar polygon; find the ratio of the areas of the polygons.
- If two polygons are similar and the length of a side of one is two-thirds of the length of the corresponding side of the other, what is the ratio of the area of the smaller polygon to the area of the larger polygon?
- If each dimension of a rectangular photograph is doubled to make an enlargement, compare the area of the enlargement with the area of the original photograph.
- In triangle ABC , D is the midpoint of \overline{AB} and E is the midpoint of \overline{AC} . Compare the areas of triangle ADE and triangle ABC .

6. In two similar polygons, the lengths of two corresponding sides are 4 and 20. If the area of the smaller polygon is 60, find the area of the larger one.
7. Find the ratio of the lengths of a pair of corresponding sides in two similar polygons if their areas are:
a. 25 and 144 *b.* 64 and 400 *c.* 50 and 288 *d.* 108 and 243
8. The areas of two similar polygons are 108 and 192. If the length of a side of the larger polygon is 8, find the length of the corresponding side of the smaller one.
9. If the lengths of two corresponding sides of similar polygons are in the ratio 1:3, then the area of the larger polygon is _____ times the area of the smaller polygon.
10. Find the ratio of the areas of two similar polygons the lengths of whose corresponding sides are in the ratio 2:3.
11. If the ratio of the lengths of two corresponding sides of two similar polygons is 1:2, express the ratio of their areas.
12. If the areas of two similar polygons are in the ratio 4:9, then the lengths of any two corresponding sides of the polygons are in the ratio _____.
13. If each angle of a polygon is kept constant and the length of each side is multiplied by 4, by what number is the area of the polygon multiplied?
14. In two similar polygons, the lengths of two corresponding sides are 4 and 2. If the area of the smaller polygon is 18 less than the area of the larger one, find the area of the larger one.
15. The area of the larger of two similar polygons is 25 times the area of the smaller one. If the length of a side of the larger polygon is 12 more than the length of the corresponding side of the smaller one, find the length of the side of the smaller polygon.
16. Find the ratio of the areas of two similar polygons if the ratio of their perimeters is:
a. 4:1 *b.* 9:1 *c.* 9:4 *d.* 2:1
17. Find the ratio of the perimeters of two similar polygons if the ratio of their areas is:
a. 16:1 *b.* 1:4 *c.* 9:25 *d.* 3:1

13. Completion Exercises

Write a word or expression that, when inserted in the blank, will make the resulting statement true.

1. If the base of the rectangle is 10 and the height is represented by $2x$, its area is represented by _____.

2. If a side of a square is represented by $3x$, its area is represented by _____.
3. If the legs of a right triangle are represented by 5 and $2x$, its area is represented by _____.
4. If a side of a rhombus is represented by x and an angle of the rhombus contains 30° , its area is represented by _____.
5. If an altitude of a trapezoid is 8 and the lengths of the bases are represented by x and $x - 4$, its area is represented by _____.
6. If the perimeter of an equilateral triangle is represented by $6x$, its area is represented by _____.
7. Two triangles are equal in area if they have congruent altitudes and congruent _____.
8. Two triangles are equal in area if they have a common base and their vertices lie on a line _____ to the base.
9. A median divides a triangle into two _____ triangles.
10. Areas of rectangles which have congruent altitudes have the same ratio as _____.
11. The diagonals of a parallelogram divide the parallelogram into four _____ triangles.
12. The base of a triangle is divided into four congruent parts. If each point of division is joined to the opposite vertex, the four triangles thus formed are _____.
13. If two triangles are similar and the area of one is four times the area of the other, the length of a side of the larger triangle is _____ times the length of the corresponding side of the smaller.
14. If two triangles are similar and the length of a side of one is four times the length of the corresponding side of the other, the area of the larger triangle is _____ times the area of the smaller one.
15. In parallelogram $ABCD$ if diagonals \overline{AC} and \overline{BD} are drawn, triangle ABC is equal in area to triangle _____.
16. The ratio of the _____ of two similar polygons is equal to the ratio of the squares of the lengths of any two corresponding sides.
17. If a triangle is equal in area to a rectangle and the base of the rectangle is congruent to the base of the triangle, then the length of the altitude of the triangle is _____ times the length of the altitude of the rectangle.
18. If the length of the base of a parallelogram is doubled and the length of the altitude is tripled, the area is multiplied by _____.
19. The length of side \overline{AB} of $\triangle ABC$ is 5 inches and the length of side \overline{AC} is 6 inches. If the number of degrees contained in angle A varies, then the largest possible area of $\triangle ABC$ is _____.

20. The diagonals of parallelogram $ABCD$ intersect in point P . Triangles APD and DPC must be _____.

14. True-False Exercises

If the statement is always true, write *true*; if the statement is not always true, write *false*.

1. If two rectangles have equal perimeters, they must have equal areas.
2. The area of a parallelogram whose angles are not right angles is equal to the product of the lengths of two of its consecutive sides.
3. Two parallelograms are always congruent if they have congruent bases and congruent altitudes.
4. The ratio of the areas of two similar triangles is equal to the ratio of the lengths of two corresponding sides.
5. Two equilateral triangles having equal perimeters must have equal areas.
6. An altitude divides a triangle into two triangles which are equal in area.
7. The area of a quadrilateral that has two parallel sides is equal to one-half the product of the sum of the lengths of the two parallel sides and the distance between them.
8. The ratio of the lengths of any two corresponding sides in two similar triangles is equal to the square root of the ratio of the areas.
9. The diagonals of a parallelogram divide it into four equivalent triangles.
10. A diagonal divides a trapezoid into two triangles whose areas have the same ratio as the lengths of the bases of the trapezoid.
11. If the lengths of the diagonals of a rhombus are represented by $4x$ and $6x$, the area is represented by $6x^2$.
12. The areas of rectangles which have congruent bases have the same ratio as the lengths of their altitudes.
13. If the area of a square is represented by $16x^2$, its side is represented by $4x$.
14. If the perimeters of two equilateral quadrilaterals are equal, their areas are equal.
15. If the length of each side of a triangle is doubled, the area of the triangle is doubled.
16. If two parallelograms have their corresponding sides congruent, their areas are equal.
17. If the area of the enlargement of a triangle is nine times the area of the triangle, then the length of a side in the original triangle is one-third the length of the corresponding side in the enlargement.

18. One angle of a rhombus contains 30° . If one of its sides is represented by x , its area is represented by $\frac{1}{2}x^2$.
19. A line segment which joins the midpoints of two sides of a triangle cuts off a triangle whose area is one-fourth the area of the given triangle.
20. If the lengths of two adjacent sides of a parallelogram remain unchanged and the included angle increases from 0° to 90° , the area of the parallelogram increases.

15. "Always, Sometimes, Never" Exercises

If the blank space in each of the following exercises is replaced by the word *always*, *sometimes*, or *never*, the resulting statement will be true. Select the word which will correctly complete each statement.

1. If two polygons are congruent, they are _____ equal in area.
2. Two rectangles of equal area _____ have unequal perimeters.
3. Triangles that have congruent bases and congruent altitudes are _____ congruent.
4. A diagonal of a parallelogram _____ divides it into two equivalent triangles.
5. If the length of a side of a rhombus none of whose angles is a right angle is represented by x and the length of the side of a square is represented by x , the area of the rhombus is _____ equal to the area of the square.
6. A median of a triangle _____ divides the triangle into two triangles which are equal in area.
7. If the dimensions of a rectangle are doubled, then its area is _____ doubled.
8. The area of a rhombus is _____ equal to one-half the product of the lengths of its diagonals.
9. The ratio of the areas of two triangles having congruent altitudes is _____ equal to the ratio of the lengths of their bases.
10. If the length of the base of one triangle is represented by $2x$ and the length of the base of another triangle is represented by $4x$, the areas of the triangles are _____ equal.
11. If the ratio of the areas of two similar triangles is 1:16, then the length of a side in the larger triangle is _____ 16 times the length of the corresponding side of the smaller triangle.
12. If two equivalent triangles have bases that are not congruent, they _____ have congruent altitudes.

13. Two triangles are _____ equal in area if their corresponding sides are equal in length.
14. If two parallelograms have bases that are unequal in length, their areas are _____ unequal.
15. If two consecutive sides of a rectangle are congruent to two consecutive sides of a parallelogram which is not a rectangle, the area of the rectangle is _____ equal to the area of the parallelogram.
16. A line segment which joins the midpoints of two adjacent sides of a parallelogram _____ cuts off a triangle which is equal in area to one-eighth of the area of the parallelogram.
17. An angle bisector in a scalene triangle _____ divides the triangle into two triangles which are equal in area.
18. A diagonal of a trapezoid _____ divides the trapezoid into two equivalent triangles.
19. If the corresponding sides of two rhombuses are congruent, their areas are _____ equal.
20. If the lengths of two consecutive sides of a parallelogram are unchanged and the measure of the included angle decreases from 90° to 0° , then the area of the parallelogram _____ increases.

16. Multiple-Choice Exercises

Write the letter preceding the word or expression that best completes the statement.

1. A median of a triangle divides it into two triangles which are always
(a) congruent (b) similar (c) equal in area.
2. The area of a rhombus is equal to (a) one-half the sum of the lengths of its diagonals (b) one-half the product of the lengths of its diagonals (c) the product of the lengths of its diagonals.
3. If the areas of two similar triangles are in the ratio 1:4, then the lengths of any two corresponding sides of these triangles are in the ratio (a) 1:4 (b) 1:2 (c) 1:16.
4. If an angle of a rhombus contains 150° and the length of one side is represented by $2m$, the area is represented by (a) $4m^2$ (b) $2m^2$ (c) $2m$.
5. The diagonals of a parallelogram divide it into four triangles whose common vertex is the intersection of the diagonals. These four triangles are always (a) congruent (b) similar (c) equal in area.
6. If the lengths of the corresponding sides of two similar triangles are in the ratio 1:2, the areas of the two triangles are in the ratio (a) 1:4 (b) 1:2 (c) $1:\sqrt{2}$.

7. Every triangle is divided into two triangles that are equal in area by (a) a median (b) the bisector of one of its angles (c) an altitude.
8. A triangle and a parallelogram having the same base are equal in area. The length of the altitude of the triangle is (a) one-fourth the length of the altitude of the parallelogram (b) one-half the length of the altitude of the parallelogram (c) twice the length of the altitude of the parallelogram.
9. If each of the dimensions of a rectangle is multiplied by 3, the area is multiplied by (a) 3 (b) 9 (c) $\sqrt{3}$.
10. If two adjacent sides of a rectangle are congruent to two adjacent sides of a parallelogram which is not a rectangle, the area of the rectangle is (a) greater than the area of the parallelogram (b) equal to the area of the parallelogram (c) smaller than the area of the parallelogram.
11. In two triangles which are equal in area, if the ratio of the lengths of the bases is 2:1, then the ratio of the length of the altitude of the first triangle to the length of the altitude of the second triangle is (a) 2:1 (b) 4:1 (c) 1:2.
12. If the lengths of two adjacent sides of a triangle are unchanged and the degree measure of the included angle increases from 0 to 90, the area of the triangle (a) increases (b) decreases (c) remains unchanged.
13. If the length of each side of a triangle is multiplied by 2, then the area of the triangle is multiplied by (a) 2 (b) 4 (c) 6.
14. If the area of a parallelogram is unchanged and the length of the altitude increases, then the length of the base (a) increases (b) decreases (c) remains unchanged.
15. If the length of the base of a triangle is doubled and the length of the altitude to the base is halved, then the area of the triangle (a) remains unchanged (b) increases (c) decreases.
16. If two triangles have a common base and their vertices lie on a line parallel to the base, the triangles are always (a) congruent (b) similar (c) equivalent.
17. If the area of the larger of two similar triangles is twice the area of the smaller triangle, then the length of each side of the larger triangle is equal to the length of the corresponding side of the smaller triangle multiplied by (a) $\sqrt{2}$ (b) 4 (c) 2.
18. "The area of a rectangle is equal to the product of the length of its base and the length of its altitude" is (a) a theorem (b) a postulate (c) a corollary.
19. If the length of each diagonal of a square is represented by d , then the area of the square is represented by (a) d^2 (b) $2d^2$ (c) $\frac{1}{2}d^2$.
20. In trapezoid $ABCD$, diagonals \overline{AC} and \overline{BD} intersect at E . The length of base \overline{AB} is 20 and the length of base \overline{DC} is 5. The ratio of the area of $\triangle DEC$ to the area of $\triangle AEB$ is (a) 1:2 (b) 1:4 (c) 1:16.

17. Construction Exercises

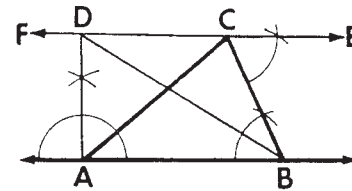
An *area-preserving transformation* transforms a given polygon into a polygon which has the same area as the given polygon.

In the following exercises, any transformation that is to be performed is to be an area-preserving transformation. A sample area-preserving transformation is shown in the following boxed problem:

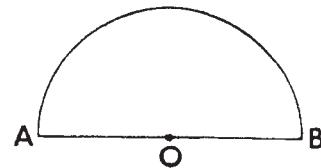
Transform a given triangle ABC into a right triangle one of whose legs will be congruent to \overline{AB} .

Solution:


1. Through C , construct $\overleftrightarrow{FE} \parallel \overleftrightarrow{AB}$.
2. At A , construct $\overleftrightarrow{AD} \perp \overleftrightarrow{AB}$.
3. Draw \overline{DB} . Triangle BAD is the required right triangle with leg \overline{AB} .
4. Since $\triangle ABC$ and BAD have a common base \overline{AB} and their vertices C and D lie on a line parallel to the base, they are equal in area.



1. Transform a given triangle ABC whose base is \overline{BC} into an isosceles triangle RBC whose base is also \overline{BC} .
2. Transform a given triangle ABC whose base is \overline{BC} into another triangle PBC whose base is \overline{BC} and which has a second side congruent to a given line segment m , the length of m being greater than the length of the altitude from A to side \overline{BC} .
3. Transform a given triangle ABC whose base is \overline{BC} into another triangle PBC whose base is \overline{BC} and which has a given angle adjacent to \overline{BC} .
4. Transform a given parallelogram $ABCD$ whose base is \overline{AB} into a rectangle whose base is \overline{AB} .
5. Transform a given parallelogram $ABCD$ whose base is \overline{AB} into a parallelogram whose base is \overline{AB} and which contains a 30° angle.
6. Transform a given parallelogram $ABCD$ whose base is \overline{AB} into a rhombus whose base is \overline{AB} .
7. Transform a given triangle ABC into an equivalent isosceles triangle which will have a given line segment m as its base.
8. Using the diameter \overline{AB} of the semicircle as a base, inscribe in the semicircle a triangle whose area shall be greater than the area of any other triangle that can be inscribed in the semicircle.



Ex. 8

9. Transform a quadrilateral into a triangle.
10. Transform a parallelogram into a triangle.
11. Transform a rectangle into a triangle.
12. Transform a trapezoid into a triangle.
13. *a.* Transform a quadrilateral into a triangle.
b. Transform the triangle found in part *a* into an isosceles triangle.
14. Transform a trapezoid into an isosceles triangle.
15. Construct a right triangle equal in area to a given right triangle and having a given line segment m as one of its legs.
16. Construct an isosceles triangle which will have a given line segment m as its base and which will be equal in area to a given parallelogram $ABCD$.
17. Construct a rectangle which will have a given line segment m as its base and which will be equal in area to a given parallelogram $ABCD$.
18. Given a right triangle whose legs are line segments a and b , transform the triangle into a rectangle whose base is a given line segment m .
19. Transform a given rectangle into a square.
20. Transform a given parallelogram into a square.
21. Construct one side of a square whose area will be equal to the area of a given triangle.
22. Construct a square equal in area to twice the area of a given triangle.
23. Construct a square equal in area to one-half the area of a given triangle.
24. It is required to construct a square equal in area to a rhombus whose diagonals are the given line segments of length d and d' .  (a) Representing the length of a side of the square by x , write an equation showing the relationship between x , d , and d' . (b) Construct x . (c) Construct the required square.
Ex. 24
25. Construct a square equal in area to the sum of the areas of two given squares.
26. Construct a square equal in area to the difference of the areas of two given squares.
27. Construct a triangle similar to two given similar triangles and equal in area to the sum of the areas of the two given triangles.
28. Construct a triangle similar to a given triangle and having twice the area of the given triangle.