CHAPTER VIII

Regular Polygons and the Circle

1. Fundamental Relationships in Regular Polygons

Let us recall some of the things we have already learned about regular polygons.

A polygon is a *regular polygon* if all its sides are congruent and all its angles are congruent, that is, if it is equilateral and equiangular.

In polygon $ABCDEF$ (Fig. 8–1), if $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE} \cong \overline{EF} \cong \overline{FA}$ and $\angle A \cong \angle B \cong \angle C \cong \angle D \cong \angle E \cong \angle F$, then polygon $ABCDEF$ is a regular polygon.

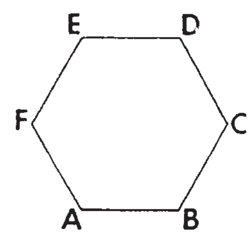


Fig. 8–1

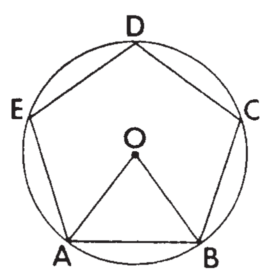


Fig. 8–2

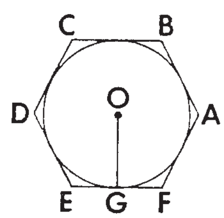


Fig. 8–3

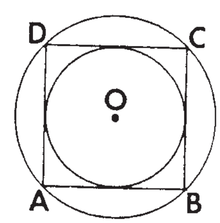


Fig. 8–4

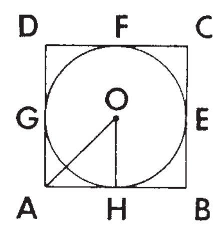


Fig. 8–5

We have learned that a *circle is circumscribed about a polygon* if the circle passes through every vertex of the polygon.

In Fig. 8–2, circle O is circumscribed about polygon $ABCDE$. We also say that polygon $ABCDE$ is *inscribed* in circle O.

***Theorem 111.* A circle may be circumscribed about any regular polygon.**

In Fig. 8–2, if $ABCDE$ is a regular polygon, then a circle whose center is at O can be circumscribed about the polygon.

***Definition.* A *radius of a regular polygon* is a radius of the circumscribed circle.**

In Fig. 8–2, a radius of regular polygon $ABCDE$ is $\overline{OA}$. Since radii of the same circle are congruent, the radii of a regular polygon are congruent. Thus, $\overline{OA} \cong \overline{OB}$.

***Definition.* A *central angle of a regular polygon* is an angle formed by two radii of the polygon drawn to consecutive vertices of the polygon.**

In Fig. 8–2, angle AOB is a central angle of regular polygon $ABCDE$.

We have learned that a *circle is inscribed in a polygon* if every side of the polygon is tangent to the circle.

In Fig. 8–3, circle O is inscribed in polygon $ABCDEF$. We also say that polygon $ABCDEF$ is circumscribed about circle O.

***Theorem 112.* A circle may be inscribed in any regular polygon.**

In Fig. 8–3, if $ABCDEF$ is a regular polygon, then a circle whose center is at O can be inscribed in the polygon.

***Definition.* An *apothem of a regular polygon* is a radius of its inscribed circle.**

In regular polygon $ABCDEF$ (Fig. 8–3), $\overline{OG}$ is the apothem. Since radii of the same circle are congruent, the apothems of a regular polygon are congruent.

***Definition.* The *center of a regular polygon* is the common center of the circumscribed and inscribed circles.**

In Fig. 8–4, if O is the common center of the circles circumscribed about and inscribed in regular polygon $ABCD$, then O is the center of regular polygon $ABCD$.

***Theorem 113.* An apothem of a regular polygon is the perpendicular bisector of the side of the polygon to which it is drawn.**

In Fig. 8–5, if $\overline{OH}$ is an apothem of regular polygon $ABCD$, then $\overline{OH}$ is perpendicular to $\overline{AB}$ and $\overline{OH}$ bisects $\overline{AB}$.

***Theorem 114.* A radius of a regular polygon bisects the angle of the polygon to whose vertex it is drawn.**

In Fig. 8–5, $\overline{OA}$, a radius of regular polygon $ABCD$, bisects angle A, the angle of the polygon to whose vertex it is drawn.

***Theorem 115.* The measure of each central angle of a regular polygon of n sides is $\dfrac{360}{n}$.**

Theorem 116. **The measure of each interior angle of a regular polygon of n sides is $\frac{(n-2)180}{n}$.**

Theorem 117. **The measure of each exterior angle of a regular polygon of n sides is $\frac{360}{n}$.**

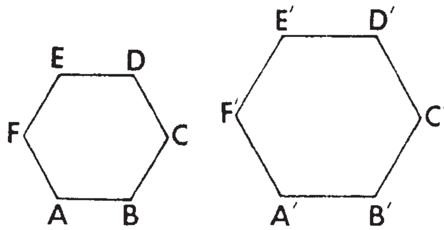


Fig. 8–6

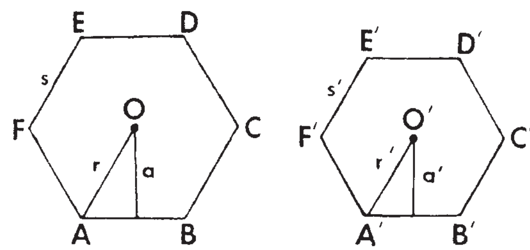


Fig. 8–7

Theorem 118. **Regular polygons of the same number of sides are similar.**

If regular polygon ABC . . . and regular polygon $A'B'C'$ . . . have the same number of sides, polygon ABC . . . ~ polygon $A'B'C'$. . . .

Note that "polygon ABC . . ." is used to represent an n-gon, which is a polygon with n sides.

In Fig. 8–6, for example, regular hexagon $ABCDEF$ ~ regular hexagon $A'B'C'D'E'F'$.

Theorem 119. **The ratio of the perimeters of regular polygons of the same number of sides is equal to the ratio of the lengths of their sides, or the ratio of the lengths of their radii, or the ratio of the lengths of their apothems.**

In Fig. 8–7, if regular polygon $ABCDEF$ and regular polygon $A'B'C'D'E'F'$ have the same number of sides, and their perimeters are represented by p and p', then

$$\frac{p}{p'} = \frac{s}{s'} \quad \text{and} \quad \frac{p}{p'} = \frac{r}{r'} \quad \text{and} \quad \frac{p}{p'} = \frac{a}{a'}$$

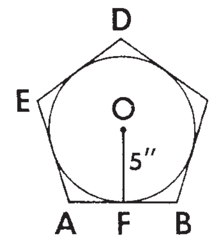
KEEP IN MIND

In proving theorems and relationships involving regular polygons, it may be helpful to circumscribe a circle about the regular polygon or to inscribe a circle in the regular polygon.

MODEL PROBLEMS

1. If the length of the apothem of a regular pentagon is 5 inches, find the length of the diameter of the circle that is inscribed in the regular pentagon.

Solution:

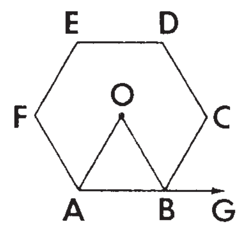


1. Since the apothem of regular pentagon $ABCDE$ is the radius of the inscribed circle, then $\overline{OF}$, the radius of circle O, is 5 inches in length.
2. The length of the diameter of circle $O = 2 \times OF = 2(5) = 10$.

Answer: The length of the diameter of the inscribed circle is 10 inches.

2. For a regular polygon of six sides, find the number of degrees contained in (*a*) each central angle (*b*) each interior angle (*c*) each exterior angle.

Solution:



a. 1. The measure of a central angle of a regular polygon of n sides $= \dfrac{360}{n}$.

2. $m\angle AOB = \dfrac{360}{6} = 60. \qquad n = 6.$

Answer: Each central angle contains 60°.

b. 1. The measure of an interior angle of a regular polygon of n sides $= \dfrac{(n-2)180}{n}$.

2. The measure of interior angle $BCD = \dfrac{(6-2)180}{6} = \dfrac{4(180)}{6} = 120.$

Answer: Each interior angle contains 120°.

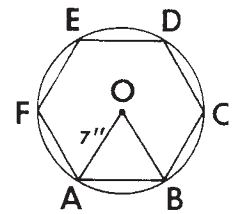
c. 1. The measure of an exterior angle of a regular polygon of n sides $= \dfrac{360}{n}$.

2. The measure of exterior angle $CBG = \dfrac{360}{6} = 60.$

Answer: Each exterior angle contains 60°.

3. A regular hexagon is inscribed in a circle. If the length of the radius of the circle is 7 in., find the perimeter of the hexagon.

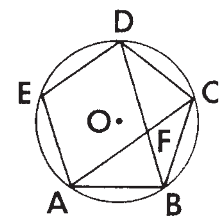
Solution:



1. Since $ABCDEF$ is a regular hexagon, the measure of central $\angle AOB = \frac{360}{6} = 60$.
2. Since $\overline{OA} \cong \overline{OB}$, $m\angle OAB = m\angle OBA = 60$.
3. Since $\triangle AOB$ is equiangular, $\triangle AOB$ is also equilateral.
4. Since the length of radius $\overline{OA}$ is 7 in., the length of side $\overline{AB}$ of the regular hexagon is 7 in.
5. Perimeter of regular hexagon $ABCDEF = 6 \times AB = 6 \times 7 = 42$.

Answer: Perimeter of the regular hexagon = 42 inches.

4. In regular polygon $ABCDE$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at F. *Prove:* $AF \times FC = BF \times FD$.



Given: $ABCDE$ is a regular polygon with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at F.

To prove: $AF \times FC = BF \times FD$.

Plan: Circumscribe circle O about regular polygon $ABCDE$. Prove that $AF \times FC = BF \times FD$ by showing that the product of the lengths of the segments of chord $\overline{AC}$ equals the product of the lengths of the segments of chord $\overline{BD}$.

Proof:

Statements	*Reasons*
1. $ABCDE$ is a regular polygon with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at F.	1. Given.
2. Circumscribe circle O about regular polygon $ABCDE$.	2. A circle may be circumscribed about any regular polygon.
3. In $\odot O$, $AF \times FC = BF \times FD$.	3. If two chords intersect inside a circle, the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other chord.

EXERCISES

1. Name the regular polygon that is (*a*) a triangle (*b*) a quadrilateral.
2. Is a rectangle a regular polygon? Why?
3. Is a rhombus a regular polygon? Why?
4. Find the measure of each central angle of a regular polygon which has the given number of sides:
   a. 3 *b.* 4 *c.* 5 *d.* 6 *e.* 8 *f.* 10
5. Find the number of degrees contained in each interior angle of a regular:
   a. triangle *b.* quadrilateral *c.* hexagon
   d. pentagon *e.* decagon *f.* octagon
6. Find the number of degrees contained in each exterior angle of a regular:
   a. quadrilateral *b.* triangle *c.* pentagon
   d. hexagon *e.* octagon *f.* decagon
7. Can an angle of 50° be a central angle of a regular polygon? Why?
8. Can an angle of 35° be an exterior angle of a regular polygon? Why?
9. If a regular hexagon is inscribed in a circle and a tangent to the circle is drawn at one of the vertices, find the number of degrees contained in the acute angle formed by the tangent and a side of the hexagon.
10. If the length of each side of a regular hexagon is represented by $2x$, represent the perimeter in terms of x.
11. The formula for the perimeter p of a regular polygon in terms of the number of sides n and the length of each side s is $p =$ _______.
12. A regular hexagon is inscribed in a circle. If the radius of the circle is 4 inches in length, a side of the hexagon is _______ inches in length.
13. A regular hexagon is inscribed in a circle. If the length of the radius of the circle is 2 inches, the perimeter of the hexagon is _______ inches.
14. The perimeter of a regular hexagon is 24. Find the length of the diameter of the circle which circumscribes this hexagon.
15. A regular hexagon is inscribed in a circle. If the length of the radius of the circle is represented by r, represent the perimeter of the hexagon in terms of r.
16. If the length of the apothem of a regular polygon is 8, what is the length of the diameter of the inscribed circle?
17. An equilateral triangle is inscribed in a circle whose radius is 8 in. long. Find the length of its apothem.
18. A square is inscribed in a circle whose radius is 12 in. long. Find the length of its apothem. [Answer may be left in radical form.]
19. A regular hexagon is inscribed in a circle the length of whose radius is 4. Find the length of its apothem. [Answer may be left in radical form.]

20. Find the ratio of the length of the apothem of a square to the length of a side of the square.

21. If the length of an apothem of a regular hexagon is $6\sqrt{3}$, find the length of a side of the hexagon.

22. The length of the apothem of a regular hexagon is represented by $4x\sqrt{3}$. Represent the perimeter of the hexagon in terms of x.

23. Two regular hexagons have perimeters of 60 and 90 inches. Find the ratio of the lengths of their apothems.

24. The ratio of the lengths of the apothems of two regular pentagons is 1:3. The perimeter of the larger polygon is how many times the perimeter of the smaller polygon?

25. *Prove:* If a circle is divided into three congruent arcs, the chords of these arcs form an inscribed regular polygon.

26. *Prove:* The length of a radius of the circle inscribed in an equilateral triangle is one-third the length of an altitude of the triangle.

27. *Prove:* An interior angle of a regular polygon is supplementary to a central angle of the polygon.

28. *Prove:* If two diagonals are drawn from a vertex of a regular pentagon, they trisect the angle at that vertex of the pentagon.

29. $ABCDE$ is a regular pentagon with diagonals $\overline{AC}$, $\overline{AD}$, and $\overline{BD}$ drawn. Prove that triangles ABD and DCA are congruent.

30. *Prove:* In regular hexagon $ABCDEF$, diagonal $\overline{AD}$ is a diameter of the circumscribed circle.

31. *Prove:* In a regular pentagon, if two diagonals intersect in the interior, the longer segment of each diagonal is congruent to a side of the pentagon.

32. *Prove:* If diagonals $\overline{EB}$ and $\overline{AD}$ of regular pentagon $ABCDE$ intersect at F, $FBCD$ is a rhombus.

33. *Prove:* In regular hexagon $ABCDEF$, if diagonals $\overline{AC}$ and $\overline{DF}$ are drawn, quadrilateral $ACDF$ is a rectangle.

34. *Prove:* In regular hexagon $ABCDEF$, if diagonals $\overline{AD}$ and $\overline{CE}$ intersect at G, then $AG \times GD = CG \times GE$.

35. In a quadrilateral, one of whose angles is a right angle, the perimeter is 80. The lengths of the sides are represented by $4x$, $5x - 5$, $3x + 5$, and $6x - 10$. (*a*) Find the length of each side of the quadrilateral. (*b*) Prove that the quadrilateral is a regular polygon.

36. In an equilateral pentagon, the measures of the angles are represented by $3x - 12$, $x + 68$, $4x - 52$, $2x + 28$, and $5x - 92$. (*a*) Find the measure of each angle of the pentagon. (*b*) Prove that the polygon is a regular polygon.

37. If the length of a side of a regular pentagon is represented by $2s$, the length of the apothem by r, and the length of the radius of the circumscribed circle by R, prove:

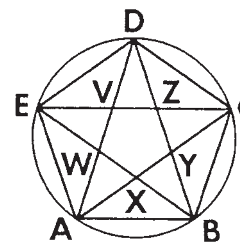
a. $r = R \cos 36°$ *b.* $s = r \tan 36°$ *c.* $s = R \sin 36°$

38. If the length of a side of a regular decagon is represented by s, the length of the apothem by r, and the length of the radius of the circumscribed circle by R, prove:

a. $r = R \cos 18°$ *b.* $s = 2r \tan 18°$ *c.* $s = 2R \sin 18°$

39. R represents the length of the radius of a circle circumscribed about a regular decagon. Prove that the perimeter of the decagon is represented by $20R \sin 18°$.

40. In the figure, $ABCDE$ is a regular pentagon inscribed in the circle. Diagonals $\overline{AC}$, $\overline{AD}$, $\overline{BD}$, $\overline{BE}$, and $\overline{CE}$ are drawn, determining points V, W, X, Y, and Z. Prove that polygon $VWXYZ$ is a regular pentagon.



Ex. 40

2. The Area of a Regular Polygon

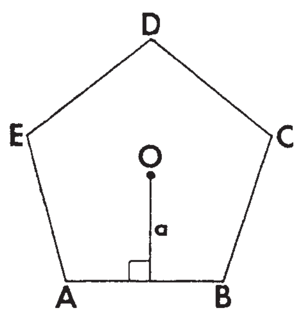


Fig. 8–8

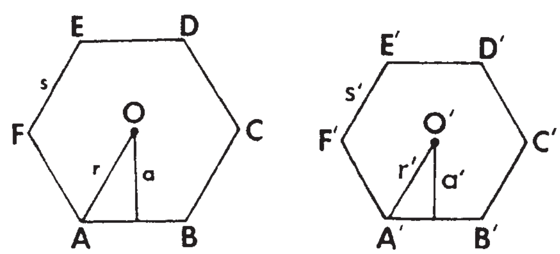


Fig. 8–9

Theorem 120. **The area of a regular polygon is equal to one-half the product of its perimeter and the length of its apothem.**

[The proof for this theorem appears on pages 769–770.]

In Fig. 8–8, if $ABCDE$ is a regular polygon the length of whose apothem is represented by a, whose perimeter is represented by p, and whose area is represented by A, then the area of polygon $ABCDE$ is given by the formula $A = \frac{1}{2}ap$.

Note. In the statement of theorem 120, we might have used the word *apothem* to refer to the "length of its apothem," which is a number. In the future, we may use this abbreviated language in situations where there can be no confusion.

Theorem 121. **The ratio of the areas of regular polygons of the same number of sides is equal to the ratio of the squares of the lengths of their sides, or the squares of the lengths of their radii, or the squares of the lengths of their apothems.**

In Fig. 8–9, if regular polygons $ABCDEF$ and $A'B'C'D'E'F'$ have the same number of sides and their areas are represented by A and A', then

$$\frac{A}{A'} = \frac{(s)^2}{(s')^2} \quad \text{and} \quad \frac{A}{A'} = \frac{(r)^2}{(r')^2} \quad \text{and} \quad \frac{A}{A'} = \frac{(a)^2}{(a')^2}$$

MODEL PROBLEMS

1. If the length of a side of a regular polygon of 12 sides is represented by s and the length of its apothem is represented by a, represent the area of the polygon, A, in terms of a and s.

Solution:

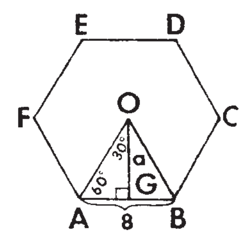
1. Since the regular polygon has 12 sides, the perimeter, $p = 12s$.

2. $A = \frac{1}{2}ap = \frac{1}{2}(a)(12s) = 6as$.

Answer: Area $= 6as$.

2. A side of a regular hexagon is 8 inches in length.
a. Find the length of the apothem of the hexagon.
b. Find the area of the hexagon. [Answers may be left in radical form.]

Solution:



a. 1. Triangle AOB is an equilateral triangle.

2. The apothem whose length is represented by a is an altitude in the equilateral triangle.

3. $a = \dfrac{\text{length of side}}{2}\sqrt{3} = \dfrac{8}{2}\sqrt{3} = 4\sqrt{3}$.

Answer: Apothem $= 4\sqrt{3}$ in.

b. 1. Perimeter of the hexagon, $p = 6 \times 8 = 48$.

2. Area of the hexagon, $A = \frac{1}{2}ap$

3. $A = \frac{1}{2}(4\sqrt{3})(48)$

4. $A = (4\sqrt{3})(24)$

5. $A = 96\sqrt{3}$

Answer: Area $= 96\sqrt{3}$ sq. in.

3. A side of a regular pentagon is 20 inches in length.

a. Find to the *nearest tenth of an inch* the length of the apothem of the pentagon.

b. Using the result obtained in answer to *a*, find to the *nearest ten square inches* the area of the pentagon.

Solution:

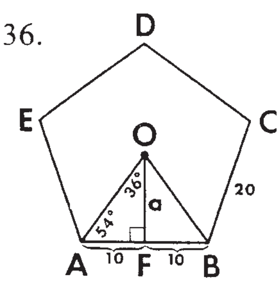
a. 1. The measure of central angle $AOB = \dfrac{360}{n}$ $\quad n = 5.$

2. $m\angle AOB = \dfrac{360}{5} = 72$

3. $m\angle AOF = \frac{1}{2}(m\angle AOB) = \frac{1}{2}(72) = 36.$

4. $m\angle OAF = 90 - m\angle AOF = 90 - 36 = 54.$

5. $\tan \angle OAF = \dfrac{\text{length of leg opposite } \angle OAF}{\text{length of leg adjacent to } \angle OAF}$



6. $\tan 54° = \dfrac{a}{10}$

7. $1.3764 = \dfrac{a}{10}$

8. $a = 10(1.3764)$

9. $a = 13.764$

10. $a = 13.8$, to the nearest tenth.

Answer: Apothem = 13.8 in.

b. 1. $p = 5(20) = 100$

2. $A = \frac{1}{2}ap$

3. $A = \frac{1}{2}(13.8)(100)$

4. $A = 690$, to the nearest ten.

Answer: Area = 690 sq. in.

NOTE. It is also possible to find the area of $\triangle AOB$ and multiply the result by 5 in order to find the area of the pentagon.

EXERCISES

1. The area K of a regular polygon the length of whose apothem is represented by a and the length of whose perimeter is represented by p is given by the formula $K =$ _______.

2. A regular polygon has a side whose length is represented by s and an apothem whose length is represented by a. Represent its area in terms of s and a if the polygon is a:
a. square *b.* hexagon *c.* decagon *d.* pentagon

3. The perimeter of a regular polygon is 24. If its apothem is 3, then its area is ______.

4. The area of a regular polygon is 144 and its perimeter is 48. Find its apothem.

5. Find the apothem and the area of a square the length of whose side is:
a. 4 *b.* 6 *c.* 8 *d.* 5 *e.* 9 *f.* $4\sqrt{2}$

6. Find, in radical form, the length of the apothem and the area of a regular hexagon the length of whose side is:
a. 2 *b.* 4 *c.* 6 *d.* 10 *e.* 5 *f.* 1

7. Find, in radical form, the apothem and the area of an equilateral triangle whose side is 12.

8. The length of a side of a regular polygon of 10 sides is 6 inches. (*a*) Find to the *nearest tenth of an inch* the length of an apothem of the polygon. (*b*) Using the result obtained in answer to *a*, find the area of the polygon to the *nearest ten square inches*.

9. The length of a side of a regular pentagon is 8. Find its area to the *nearest integer*.

10. The perimeter of a regular pentagon is 50 inches. Find its area to the *nearest ten square inches*.

11. The length of a diameter of a circle is 20 inches. (*a*) Find the length of the apothem and the area of a regular inscribed hexagon. (*b*) Find the length of a side and the area of an inscribed equilateral triangle. [Both answers may be left in radical form.]

12. Find the apothem and the area of an equilateral triangle which is circumscribed about a circle whose radius is 10. [Answer may be left in radical form.]

13. Find the apothem and the area of a regular hexagon which is circumscribed about a circle whose radius is 6. [Answer may be left in radical form.]

14. A regular pentagon is circumscribed about a circle whose radius is 10. (*a*) Find a side of the pentagon to the *nearest tenth*. (*b*) Using the result obtained in answer to *a*, find the area of the pentagon to the *nearest integer*.

15. A regular decagon (10 sides) whose side is 18 inches in length is inscribed in a circle. (*a*) Find to the *nearest tenth of an inch* the apothem of the decagon. (*b*) Using the result obtained in answer to *a*, find to the *nearest square inch* the area of the decagon.

16. (*a*) If r represents the length of the radius of a circle circumscribed about

a regular pentagon, show that the length of an apothem of the pentagon is equal to $r \cos 36°$ and its perimeter is equal to $10r \sin 36°$. (*b*) Find to the *nearest integer* the length of the apothem and the perimeter of the pentagon when $r = 4$. (*c*) Using the results found in answer to *b*, find the area of the pentagon.

17. The area of a regular hexagon is $54\sqrt{3}$. Find the length of an altitude of an equilateral triangle that has its perimeter equal to that of the hexagon.

18. An equilateral triangle is inscribed in a circle the length of whose radius is represented by r. (*a*) Show that the length of an apothem of the equilateral triangle is represented by $\frac{1}{2}r$. (*b*) Show that the area of the equilateral triangle is represented by $\frac{3}{4}r^2\sqrt{3}$.

19. A square is inscribed in a circle of radius r. (*a*) Show that the apothem of the square is represented by $\frac{1}{2}r\sqrt{2}$. (*b*) Show that the area of the square is represented by $2r^2$.

20. Find the length of a side of a regular hexagon whose area is:
a. $6\sqrt{3}$ *b.* $54\sqrt{3}$ *c.* $150\sqrt{3}$ *d.* $216\sqrt{3}$ *e.* $\frac{27}{2}\sqrt{3}$

21. *Prove:* The area of the square circumscribed about a circle equals twice the area of the square inscribed in the circle.

22. *Prove:* The area of a regular hexagon inscribed in a circle is twice the area of an equilateral triangle inscribed in the same circle.

23. The radii of two regular hexagons are 8 and 2. Find the ratio of the areas of the two polygons.

24. The areas of two regular polygons which have the same number of sides are 900 sq. in. and 100 sq. in. Find the ratio of their apothems.

25. The area of the larger of two regular polygons which have the same number of sides is 16 times the area of the smaller polygon. Find the ratio of the lengths of the radii of the two polygons.

3. The Circumference of a Circle

Observe in Fig. 8–10 that as the number of sides of a regular polygon inscribed in a circle is increased from 3 to 6 to 12 to 24, the polygon resembles the circle more and more. Yet, no matter how large the number of sides of the polygon may become, the polygon never really becomes a circle.

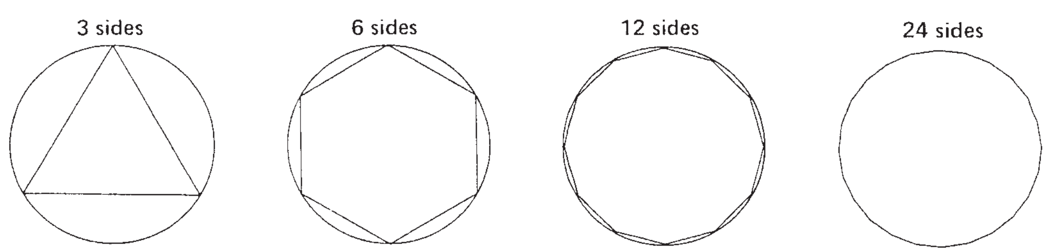


Fig. 8–10

As the number of sides of the regular polygon is increased, the perimeter of the polygon approaches the perimeter of the circle, which is called the *circumference* of the circle.

Definition. **The *circumference* of a circle is the length of the circle expressed in linear units (inches or feet).**

If the perimeter of each circle in Fig. 8–10 is 3 inches, we say that the circumference of each circle is 3 inches.

In light of the previous discussion, it appears reasonable to accept the following postulate:

Postulate 45. **When the number of sides of a regular polygon which is inscribed in a circle increases without bound, the perimeter of the polygon approaches the circumference of the circle and may be used as an approximation of the circumference of the circle.**

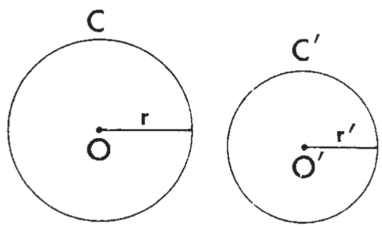


Fig. 8–11

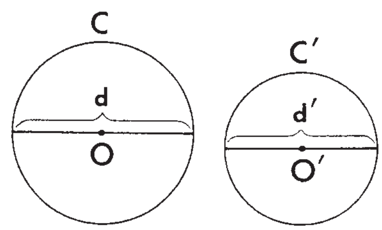


Fig. 8–12

Theorem 122. **The ratio of the circumferences of two circles is equal to the ratio of the lengths of their radii.**

In circles O and O' (Fig. 8–11), whose circumferences are represented by C and C' and the lengths of whose radii are represented by r and r', $\frac{C}{C'} = \frac{r}{r'}$.

Corollary T122–1. **The ratio of the circumferences of two circles is equal to the ratio of the lengths of their diameters.**

In circles O and O' (Fig. 8–12), whose circumferences are represented by C and C' and the lengths of whose diameters are represented by d and d', $\frac{C}{C'} = \frac{d}{d'}$.

Corollary T122–2. **The ratio of the circumference of any circle to the length of its diameter is equal to the ratio of the circumference of any other circle to the length of its diameter.**

In the two circles pictured in Fig. 8–12, $\frac{C}{d} = \frac{C'}{d'}$.

Since the ratio of the circumference of a circle to the length of its diameter is always the same no matter what the size of the circle may be, we say that this ratio is a *constant*. This constant is represented by the Greek letter π, read "pi." Therefore, we have:

$$\frac{C}{d} = \pi$$

The number π is an irrational number which cannot be expressed exactly as an integer, a terminating decimal, or a repeating decimal. The approximate value of π can be expressed to any desired degree of accuracy. The most frequently used approximate values of π are: 3.1, 3.14, 3.1416, $3\frac{1}{7}$, and $\frac{22}{7}$.

***Corollary T122–3.* The circumference, *C*, of a circle the length of whose diameter is represented by *d* and the length of whose radius is represented by *r* is given by the formula:**

$$C = \pi d \quad \text{or} \quad C = 2\pi r$$

MODEL PROBLEMS

1. Find the circumference of a circle whose radius is 21 in. [Use $\pi = \frac{22}{7}$.]

Solution:

1. $C = 2\pi r$ $\qquad r = 21,\ \pi = \frac{22}{7}$.
2. $C = 2(\frac{22}{7})(21)$
3. $C = 132$

Answer: Circumference = 132 in.

2. Find the diameter of a circle whose circumference is 628 ft. [Use $\pi =$ 3.14.]

Solution:

1. $C = \pi d$ $\qquad C = 628,\ \pi = 3.14$.
2. $628 = 3.14d$
3. $62{,}800 = 314d$
4. $200 = d$

Answer: Diameter = 200 ft.

EXERCISES

1. In any circle, what is the ratio of the circumference to the diameter exactly equal to?

2. The formula for the circumference C of a circle in terms of its radius r is $C =$ ______.

3. Represent, in terms of x, the circumference of a circle whose radius is represented by:
a. x *b.* $2x$ *c.* $3x$ *d.* $4x$ *e.* $2x + 1$

4. Represent, in terms of x, the circumference of a circle whose diameter is represented by:
a. $2x$ *b.* $4x$ *c.* x *d.* $3x$ *e.* $x + 3$

5. Use $\pi = \frac{22}{7}$ or 3.14 to find the circumference of a circle in which:
a. $r = 14$ *b.* $r = 20$ *c.* $r = 3.5$ *d.* $d = 28$ *e.* $d = 60$

6. Find the number of inches in the circumference of a circle in which the longest chord that can be drawn measures 16 inches.

7. Find the number of miles in the length of the equator if the radius of the earth is taken as 4000 miles. [$\pi = 3.14$]

8. A wagon wheel has a radius of 2.8 feet. How many feet will it travel if it makes 50 revolutions?

9. A circle is inscribed in a square whose side is 8. Find the circumference of the circle in terms of π.

10. If the circumference of a circle is 10π, find its radius.

11. Find the diameter of a circle whose circumference is 8π.

12. Find, in terms of π, the circumference of a circle which is circumscribed about a square whose side is:
a. 4 in. *b.* 5 in. *c.* 12 in.

13. Find the radius and diameter of a circle whose circumference is:
a. 40π *b.* 3.5π *c.* 44 *d.* 15.7 *e.* 25

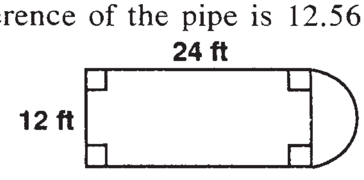
14. The circumference of a circle is increased from 30π inches to 50π inches. By how many inches is the length of the radius *increased?*

15. If the radius of a circle is increased by x, the circumference of the circle is increased by (*a*) x (*b*) $2x$ (*c*) $2\pi x$.

16. If the circumference of a circle is 88 inches, the radius of the circle is ______ inches. [$\pi = \frac{22}{7}$]

17. Find the radius of a pipe if the circumference of the pipe is 12.56 ft.

18. The swimming pool shown has a semicircular shallow end. Find the perimeter of the pool.



19. Represent, in terms of x, the radius of a circle whose circumference is represented by: *a.* $2\pi x$ *b.* $6x\pi$ *c.* $8\pi x$ *d.* $3x\pi$ *e.* $7\pi x$

20. Find the number of inches in the diagonal of a square inscribed in a circle whose circumference is:
a. 12π in. *b.* 9π in. *c.* 44 ft.

21. A square banner for a school play shows a circular logo inscribed in the banner. The radius of the circle is 3 feet. Find the circumference of the circle and the perimeter of the banner. [Use $\pi = 3.14$]

22. Points L, M, and N lie on a circle with M the midpoint of the major arc $\overset{\frown}{LN}$. The diameter through M intersects chord $\overline{LN}$ at R and minor arc $\overset{\frown}{LN}$ at P. The length of chord $\overline{LN}$ is 4 inches and $\overline{MR}$ is 3 inches longer than $\overline{RP}$. (*a*) If the length of $\overline{RP}$ is represented by x, represent the length of $\overline{MR}$ in terms of x. (*b*) Find the length of $\overline{RP}$. (*c*) Find the circumference of the circle in terms of π.

23. Quadrilateral $QRST$ is inscribed in a circle and the degree measures of arcs $\overset{\frown}{RS}$, $\overset{\frown}{ST}$, $\overset{\frown}{TQ}$, and $\overset{\frown}{QR}$ are represented by $x + 30$, $2x$, $x + 10$, and $4x - 80$ respectively. (*a*) Find the number of degrees contained in each of the four arcs. (*b*) If $\overline{RT}$ is drawn, find the number of degrees contained in angle QRT. (*c*) If the length of chord $\overline{QT}$ is 14, find the circumference of the circle. [$\pi = \frac{22}{7}$]

24. Points A, B, and C lie on a circle with B the midpoint of the major arc $\overset{\frown}{AC}$. The diameter through B intersects chord $\overline{AC}$ at D and minor arc $\overset{\frown}{AC}$ at E. $\overline{AC}$ is 8 inches in length, and $\overline{BD}$ is 6 inches longer than $\overline{DE}$.
a. If the length of $\overline{DE}$ is represented by x, express the length of $\overline{BD}$ in terms of x.
b. Which of the following equations can be used to find the length of $\overline{DE}$?
(1) $2x + 6 = 8$ (2) $x^2 + 6x = 16$ (3) $x^2 + 6x = 8$
c. Find the length of $\overline{DE}$.
d. Find the circumference of the circle. [Answer may be left in terms of π.]

25. The circumference of a circle is equal to the perimeter of a square the length of whose side is represented by s. Show that the length of the radius of this circle is equal to $\frac{2s}{\pi}$.

4. Finding the Length of an Arc

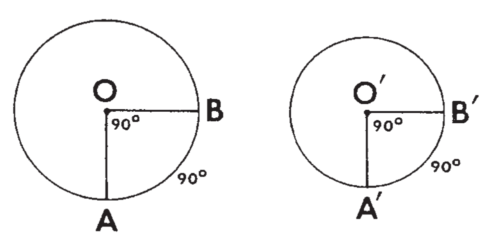


Fig. 8–13

The *length of an arc* (Fig. 8–13), which is the number of linear units it contains, is frequently confused with the number of arc degrees it contains. To say that an arc $\overset{\frown}{AB}$ contains 90 arc degrees indicates that the circle has been divided into 360 congruent parts each of which is called an *arc degree* and that arc $\widehat{AB}$ contains 90 of these arc degrees. Such an arc would be $\frac{90}{360}$ or one-fourth of the circle. In a large circle, an arc of 90° would have a greater length than an arc of 90° in a smaller circle. In circles O and O', $\widehat{AB}$ and $\widehat{A'B'}$ each contain 90°, yet $\widehat{AB}$ has a greater length than $\widehat{A'B'}$.

Theorem 123. **In a circle whose circumference is represented by C, the length of an arc which contains $n°$, or whose central angle contains $n°$, is given by the formula:**

$$\frac{\text{length of an arc}}{\text{circumference of the circle}} = \frac{n}{360} \quad \text{or} \quad \text{length of an arc} = \frac{n}{360} \times C$$

Corollary T123–1. **In a circle the length of whose radius is represented by r, the length of an arc which contains $n°$, or whose central angle contains $n°$, is given by the formula:**

$$\frac{\text{length of an arc}}{2\pi r} = \frac{n}{360} \quad \text{or} \quad \text{length of an arc} = \frac{n}{360} \times 2\pi r$$

MODEL PROBLEMS

1. Find to the *nearest tenth of an inch* the length of an arc of 60° in a circle whose radius is 12 in.

Solution:

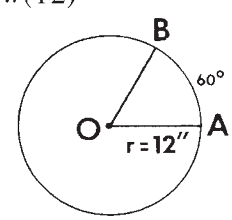
Let $l =$ length of the arc.

Method 1

1. $\frac{\text{length of arc}}{\text{circumference of circle}} = \frac{n}{360}$.
2. $\frac{l}{2\pi r} = \frac{60}{360}$ $\quad n = 60.$ $\quad r = 12.$
3. $\frac{l}{24\pi} = \frac{1}{6}$
4. $6l = 24\pi$
5. $l = 4\pi$ $\quad$ Use $\pi = 3.14$.
6. $l = 4(3.14) = 12.56$
7. $l = 12.6$ to the nearest tenth

Method 2

1. $l = \frac{n}{360} \times 2\pi r$
2. $l = \frac{60}{360} \times 2\pi(12)$

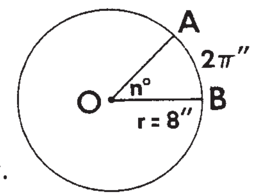


3. $l = \frac{1}{6} \times 24\pi$
4. $l = 4\pi$
5. $l = 4(3.14) = 12.56$
6. $l = 12.6$ to the nearest tenth

Answer: Length of arc = 12.6 in. to the nearest tenth of an inch.

2. In a circle whose radius is 8 inches, find the number of degrees contained in the central angle of an arc whose length is 2π inches.

Solution:



1. $\dfrac{\text{length of arc}}{\text{circumference of circle}} = \dfrac{n}{360}.$

2. Since $r = 8$, circumference of circle $= 2(\pi)(8) = 16\pi$.

3. $\dfrac{2\pi}{16\pi} = \dfrac{n}{360}$

4. $\dfrac{1}{8} = \dfrac{n}{360}$

5. $8n = 360$

6. $n = 45$

Answer: Central angle contains 45°.

EXERCISES

1. In a circle whose circumference is 36 inches, find the length of an arc which contains:
a. 30° *b.* 60° *c.* 90° *d.* 240° *e.* 45°

2. Find the circumference of a circle if the length of an arc of the circle which contains 90° is 5 inches.

3. An arc of a circle contains 72° and is 10 inches long. Find the circumference of the circle.

4. In a circle whose radius is 12, find the length of an arc whose central angle contains:
a. 60° *b.* 90° *c.* 150° *d.* 240° *e.* 50°

5. In a circle whose radius is 18, a central angle measures 80°. Find the length of the arc which the central angle intercepts.

6. In a circle whose circumference is 24 in., find the measure of the central angle of an arc whose length is 8 in.

7. In a circle whose radius is 16, find the measure of the central angle of an arc whose length is
a. 4π *b.* 8π *c.* 12π *d.* 16π *e.* 20π

8. Find the radius of a circle in which an arc which contains 120° has a length of 6π in.

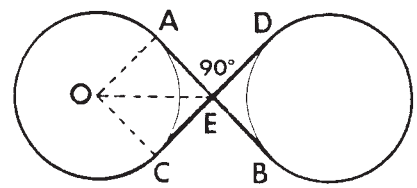
9. Find the radius of a circle in which an arc which contains 60° has the length of:
a. 3π *b.* 5π *c.* 6π *d.* 8π *e.* 12π

10. Find the length of an arc intercepted by a side of a regular hexagon inscribed in a circle whose radius is 18.

11. In a circle whose radius is 4, a chord is drawn perpendicular to one radius and bisecting that radius. (*a*) How many degrees are contained in the minor arc of this chord? (*b*) Find the length of this arc to the *nearest integer.* [$\pi = 3.14$] (*c*) Find the length of the chord to the *nearest integer.*

12. Tangents $\overline{PA}$ and $\overline{PB}$ are drawn to a circle from an external point P. $m\angle BPA = 80$ and $PA = 21$. (*a*) Find the number of degrees contained in minor arc $\overparen{AB}$. (*b*) Find to the *nearest tenth* the radius of the circle. (*c*) Find to the *nearest integer* the length of minor arc $\overparen{AB}$. [$\pi = 3.14$]

13. A belt moves over two wheels which have the same size and crosses itself at right angles as shown in the figure. A, B, C, and D are points of tangency; E is the intersection of the tangents; and O is the center of one of the circles. The radius of each wheel is 7 inches.



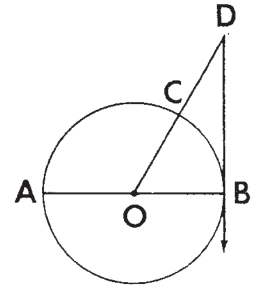
Ex. 13

a. Show that $m\angle AOE = 45$.
b. Find:
(1) the length of $\overline{AE}$.
(2) the length of major arc $\overparen{AC}$. [$\pi = \frac{22}{7}$]
(3) the length of the entire belt.

14. In a circle whose radius is 4 inches, find to the *nearest inch* the length of the minor arc intercepted by a chord 6 inches in length.

15. In a circle whose radius is 20 inches, find to the *nearest inch* the length of the minor arc intercepted by a chord 24 inches in length.

16. In circle O, $\overline{AB}$ is a diameter. Radius $\overline{OC}$ is extended to meet the tangent $\overleftrightarrow{BD}$ at D. The measures of arc $\overparen{AC}$ and arc $\overparen{CB}$ are in the ratio 2:1, and $AB = 32$.



Ex. 16

a. Find the number of degrees contained in arc $\overparen{CB}$.
b. Find to the *nearest integer* the perimeter of the figure bounded by $\overline{BD}$, $\overline{DC}$, and arc $\overparen{CB}$. [$\pi = 3.14$]

5. Finding the Area of a Circle

***Definition.* A *circular region* is the union of a circle and its interior.**

A circular region is pictured in Fig. 8–14. For convenience when we are discussing the area of a circular region, we will refer to it as the *area of a circle.*

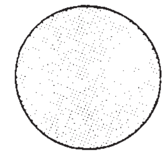


Fig. 8–14

Observe in Fig. 8–15 that as the number of sides of a regular polygon inscribed in a circle O is increased from 3 to 6 to 12, etc., the length of the apothem, a, approaches r, the length of the radius of the polygon, which is also the radius of the circle. Also, the area of the polygon approaches the area of the circle.

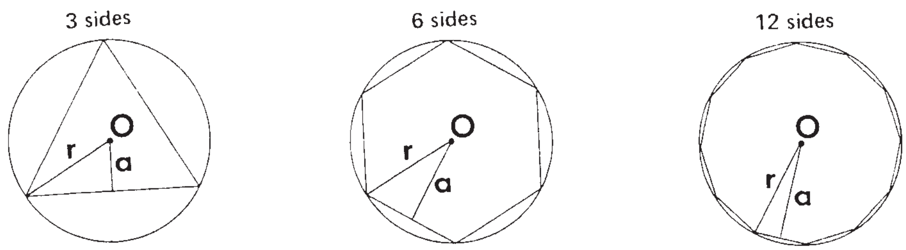


Fig. 8–15

Hence, it appears reasonable to accept the following postulate:

***Postulate 46.* When the number of sides of a regular polygon inscribed in a circle increases without bound, (*a*) the length of the apothem of the polygon approaches the length of the radius of the circle and may be used as an approximation of the length of the radius of the circle, and (*b*) the area of the regular polygon approaches the area of the circle and may be used as an approximation of the area of the circle.**

***Theorem 124.* The area of a circle is equal to one-half the product of its radius and circumference.**

If the length of the radius of a circle is represented by r and the circumference of the circle is represented by C, the area of the circle, A, is given by the formula $A = \frac{1}{2}rC$.

***Corollary T124–1.* The area of a circle is equal to π times the square of the length of the radius.**

If the length of the radius of a circle is represented by r, the area of the circle, A, is given by the formula $A = \pi r^2$.

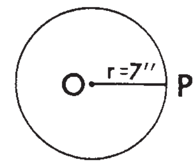
***Corollary T124–2.* The area of a circle is equal to $\frac{1}{4}\pi$ times the square of the length of the diameter.**

If the length of the diameter of a circle is represented by d, the area of the circle, A, is given by the formula $A = \frac{1}{4}\pi d^2$.

MODEL PROBLEMS

1. Find the area of a circle whose radius is 7 in. [Use $\pi = \frac{22}{7}$.]

Solution:

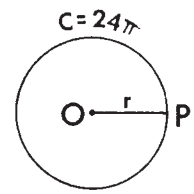


1. $A = \pi r^2$ $\qquad r = 7,\ \pi = \frac{22}{7}.$
2. $A = \pi(7)^2$
3. $A = 49\pi$
4. $A = 49 \times \frac{22}{7} = 154$

Answer: Area = 154 sq. in.

2. If the circumference of a circle is 24π ft., find the area of the circle. [Leave answer in the form of π.]

Solution:



1. $C = 2\pi r$ $\qquad C = 24\pi.$
2. $24\pi = 2\pi r$
3. $r = \dfrac{24\pi}{2\pi} = 12$
4. $A = \pi r^2$
5. $A = \pi(12)^2 = 144\pi$

Answer: Area = 144π sq. ft.

EXERCISES

1. The formula for the area A of a circle, in terms of its diameter d, is (*a*) πd^2 (*b*) $\frac{1}{4}\pi d^2$ (*c*) $\frac{1}{2}\pi d^2$.

2. Represent, in terms of x, the area of a circle whose radius is represented by:
a. $2x$ *b.* $3x$ *c.* $4x$ *d.* $8x$ *e.* $x+1$

3. Represent, in terms of x, the area of a circle whose diameter is represented by:
a. $2x$ *b.* $4x$ *c.* $10x$ *d.* $3x$ *e.* $5x$

4. Find, in terms of π, the area of a circle in which:
a. $r = 4$ *b.* $r = 8$ *c.* $r = 2\frac{1}{2}$ *d.* $d = 12$ *e.* $d = 5$

5. Find the area of a circle inscribed in a square whose side is 8. [Answer may be left in terms of π.]

6. The area of a square is 16 sq. in. Find the area of the inscribed circle. [Answer may be left in terms of π.]

7. Find the area of a circle circumscribed about a square whose apothem has a length of 1 inch.

8. Find the radius of a circle whose area is:
a. 4π *b.* 49π *c.* $.64\pi$ *d.* $\frac{9}{25}\pi$ *e.* $6\frac{1}{4}\pi$ *f.* 8

9. Represent, in terms of x, the radius of a circle whose area is represented by:
a. $4x^2\pi$ *b.* $25x^2\pi$ *c.* $\frac{64}{9}x^2\pi$ *d.* $100\pi x^2$ *e.* $\frac{1}{9}\pi x^2$

10. Find the area of a circle circumscribed about a regular hexagon whose side is:
a. 4 *b.* 12 *c.* 20 *d.* 1 *e.* 5

11. Find the area of a circle whose circumference is:
a. 16π *b.* 10π *c.* 5π *d.* $\frac{49\pi}{4}$ *e.* 10

12. Represent, in terms of x, the area of a circle whose circumference is represented by:
a. $4x\pi$ *b.* $6x\pi$ *c.* $12\pi x$ *d.* $3x\pi$ *e.* $9\pi x$

13. Find the circumference of a circle whose area is:
a. 25π *b.* 36π *c.* 100π *d.* $\frac{49}{4}\pi$ *e.* π

14. Represent, in terms of x, the circumference of a circle whose area is represented by:
a. $4x^2\pi$ *b.* $25\pi x^2$ *c.* $16\pi x^2$ *d.* $\frac{49}{4}\pi x^2$

15. The area of a circle is 81π; find the length of a side of the inscribed regular hexagon.

16. Diagonals $\overline{AC}$ and $\overline{BD}$ of rhombus $ABCD$ intersect at O, $\overline{BD}$ being the shorter diagonal. A line is drawn from O perpendicular to $\overline{AB}$, meeting it at X. $AB = BD = 12$. (*a*) Find OB, BX, and OX. (*b*) Find, in terms of π, the area of the circle that can be inscribed in the rhombus.

17. The lengths of the diagonals of a rhombus are 60 and 80. Find the (*a*) area of the rhombus (*b*) length of one side (*c*) length of the altitude of the rhombus (*d*) area of the inscribed circle. [Answer may be left in terms of π.]

18. Given a square the length of whose side is represented by s. (*a*) Express the area of the inscribed circle in terms of π and s. (*b*) Express the area of the circumscribed circle in terms of π and s. (*c*) Find, in *simplest form*, the ratio of the area of the inscribed circle to the area of the circumscribed circle.

19. Find the radius of a circle whose area is equal to the sum of the areas of two circles whose radii are 12 in. and 16 in.

20. Find the area of a circle circumscribed about a square whose side is 2.

21. Quadrilateral $ABCD$ is inscribed in circle O; and the measures of arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, and $\overset{\frown}{DA}$ are in the ratio 3:4:5:6 respectively. (*a*) Find the number of degrees contained in each arc. (*b*) Find the number of degrees contained in angle BAD. (*c*) If the length of side $\overline{AB}$ of the quadrilateral equals 10, find the area of the circle. [*Hint:* Draw $\overline{BD}$.] [Use $\pi = 3.14$.]

22. In the figure, $\overline{AB}$ and $\overline{CD}$ are parallel chords on opposite sides of the center of circle O. $CD = 12$, $AB = 16$, and RS, the distance between the chords, is 14.

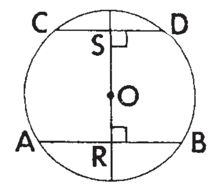
a. If the length of $\overline{OR}$ is represented by x, express the length of $\overline{OS}$ in terms of x.

b. What is the length of $\overline{RB}$? of $\overline{SD}$?

c. In terms of x, write *two* expressions for the square of the length of the radius of the circle. [*Hint:* Draw $\overline{OB}$ and $\overline{OD}$.]

d. Using the results obtained in answer to c, find the length of $\overline{OR}$.

e. Find the area of the circle. [Answer may be left in terms of π.]



Ex. 22

6. Finding the Area of a Sector of a Circle

In Fig. 8–16, the part of the circular region which is shaped like a "slice of pie" is called a *sector of a circle.*

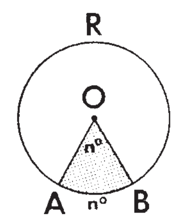


Fig. 8–16

Definition. **A *sector of a circle* is the union of an arc of the circle, the two radii which are drawn from the center of the circle to the endpoints of the arc, and the interior of the region bounded by these two radii and the arc.**

In circle O (Fig. 8–16), shaded sector AOB is a *minor sector.* Sector $OARB$ is a *major sector.* In minor sector AOB, angle AOB is referred to as the "central angle of the sector," or the angle of the sector. In this case, we would say that the degree measure of the angle of sector AOB is n.

Theorem 125. **In a circle whose area is represented by A, the area of a sector whose central angle or intercepted arc contains $n°$ is given by the formula:**

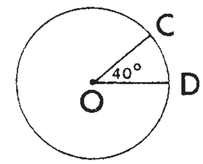
$$\frac{\text{area of a sector}}{\text{area of the circle}} = \frac{n}{360} \quad \text{or} \quad \text{area of a sector} = \frac{n}{360} \times A$$

Corollary T125–1. **In a circle the length of whose radius is represented by r, the area of a sector whose central angle or intercepted arc contains $n°$ is given by the formula:**

$$\frac{\text{area of a sector}}{\pi r^2} = \frac{n}{360} \quad \text{or} \quad \text{area of a sector} = \frac{n}{360} \times \pi r^2$$

MODEL PROBLEMS

1. The area of circle O is 36 sq. in. If there are 40° contained in central angle COD, find the number of square inches in the area of sector COD.

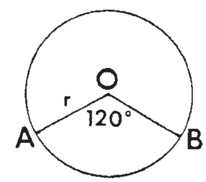


Solution:

1. Area of sector $= \dfrac{n}{360} \times A$ $\qquad n = 40.$
   A, the area of circle $O = 36.$
2. Area of sector $= \frac{40}{360} \times 36$
3. Area of sector $= \frac{1}{9} \times 36 = 4$

Answer: Area of sector = 4 sq. in.

2. In circle O, a sector whose angle contains 120° has an area of 3π sq. in. Find the radius of the circle.



Solution:

Method 1

1. $\dfrac{\text{area of sector}}{\text{area of circle}} = \dfrac{n}{360}$

   area of sector $= 3\pi.$

   $n = 120.$

2. $\dfrac{3\pi}{\pi r^2} = \dfrac{120}{360}$
3. $\dfrac{3\pi}{\pi r^2} = \dfrac{1}{3}$
4. $\pi r^2 = 9\pi$
5. $r^2 = 9$
6. $r = 3$

Method 2

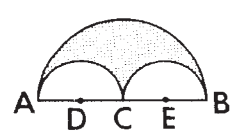
1. Since the central angle of the sector, $\angle AOB$, contains 120°, the area of the sector is $\frac{120}{360}$, or $\frac{1}{3}$ of the area of the circle.
2. Since the area of the sector is 3π, the area of the circle is $3(3\pi) = 9\pi$.
3. $\pi r^2 = 9\pi$
4. $r^2 = 9$
5. $r = 3$

Answer: Radius of the circle is 3 in.

EXERCISES

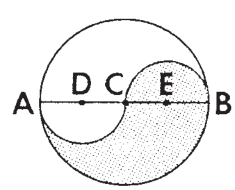
1. Find what fractional part of the area of a circle the area of a sector is if its central angle contains: *a.* 90° *b.* 30° *c.* 135° *d.* 240° *e.* 40°

2. If the central angle of a sector of a circle contains 40°, the area of the circle is ______ times the area of the sector.
3. If the area of a sector of a circle is to the area of the circle as 1:8, then the number of degrees contained in the angle of the sector is ______.
4. The area of a circle is 80 square inches. Find the area of a sector whose central angle contains 45°.
5. The area of a circle is 60 sq. ft. Find the area of a sector whose central angle measures 120.
6. The angle of a sector of a circle measures 90 and the area of the circle is 64π. Find, in terms of π, the area of the sector.
7. The angle of a sector of a circle measures 60 and the area of the circle is 144π sq. in. Find the area of the sector to the *nearest square inch.*
8. In a circle whose radius is 12, find the area of a sector whose central angle contains: *a.* 90° *b.* 45° *c.* 60° *d.* 120° *e.* 80°
9. The radius of a circle is 9 and the angle of a sector of this circle measures 40. The area of this sector, in terms of π, is ______.
10. Find the radius of a circle in which a sector whose central angle measures 90 has an area of: *a.* 4π *b.* 25π *c.* 36π *d.* 9π *e.* π
11. The angle of a sector of a circle measures 120 and the area of the sector is 27π. Find the radius of the circle.
12. The area of the sector of a circle whose angle measures 40 is 4π. Find the radius of the circle.
13. In a circle whose radius is 12, find the measure of the central angle of a sector whose area is: *a.* 36π *b.* 12π *c.* 72π *d.* 16π *e.* 60π
14. Find the number of degrees contained in the angle of a sector if its area is 5π and the radius of the circle is 6.
15. $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are tangents to circle O at A and C respectively, and form an angle of 50°. Line segments $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ are drawn. The length of $\overline{OB}$ is 14.2 inches. (*a*) Find to the *nearest inch* the length of the radius of the circle. (*b*) Using the value of the radius obtained in answer to *a* and using $\pi = \frac{22}{7}$, find to the *nearest square inch* the area of minor sector AOC.
16. $\overline{AB}$ is the diameter of the large semicircle and $\overline{AC}$ and $\overline{CB}$ are the diameters of the small semicircles. Prove that the area of the shaded region is equal to the sum of the areas of the two small semicircles.



Ex. 16

17. $\overline{AB}$ is the diameter of the large circle. $\overline{AC}$ and $\overline{CB}$ are the diameters of the small semicircles. Prove that the area of the shaded part of the large circle is equal to the area of the unshaded part of the large circle.



Ex. 17

7. Finding the Area of a Segment of a Circle

***Definition.* A *segment of a circle* is the union of an arc of the circle, its chord, and the interior of the region bounded by the arc and the chord.**

In circle O (Fig. 8–17), the shaded segment ASB is a *minor segment*. Segment ARB, the segment that is not shaded, is a *major segment*.

To find the area of a minor segment of a circle (for example, minor segment ASB in Fig. 8–18), we can use the following relationship:

Area of segment ASB = area of minor sector $OASB$ − area of triangle OAB.

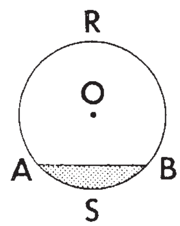


Fig. 8–17

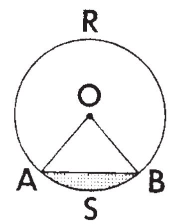


Fig. 8–18

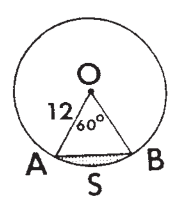
To find the area of a major segment of a circle, we subtract the area of the minor segment from the area of the circle. In Fig. 8–18:

Area of major segment ARB = area of circle O − area of minor segment ASB.

MODEL PROBLEMS

1. In a circle whose radius is 12, find the area of a minor segment whose arc has a central angle which contains 60°. [Answer may be left in radical form and in terms of π.]

Solution:

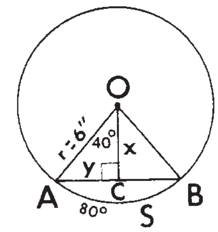


1. Area of sector $OASB = \frac{n}{360} \times \pi r^2 \quad n = 60,\ r = 12.$
2. Area of sector $OASB = \frac{60}{360} \times (12)^2\pi = \frac{1}{6}(144\pi) = 24\pi$
3. Area of equilateral triangle $OAB = \frac{s^2}{4}\sqrt{3} = \frac{(12)^2}{4}\sqrt{3}$
4. Area of equilateral triangle $OAB = \frac{144}{4}\sqrt{3} = 36\sqrt{3}$
5. Area of segment ASB = area of sector $OASB$ − area of triangle OAB
6. Area of segment $ASB = 24\pi - 36\sqrt{3}$

Answer: Area of segment $= 24\pi - 36\sqrt{3}$.

2. In a circle whose radius is 6 in., find to the *nearest square inch* the area of a segment whose chord has an arc which contains 80°. [Use $\pi = 3.14$.]

Solution:



1. Since $m\widehat{AB} = 80$, $m\angle AOB = 80$.
2. Area of sector $OASB = \frac{n}{360} \times \pi r^2$ $n = 80$. $r = 6$.
3. Area of sector $OASB = \frac{80}{360} \times (6)^2\pi = \frac{2}{9} \times 36\pi$
4. Area of sector $OASB = 8\pi = 8(3.14) = 25.12$, or 25.1

NOTE. Since the answer is to be correct to the nearest whole number, we will round off to tenths all numbers that lead to the answer.

5. To find the area of $\triangle AOB$, we will use $A = \frac{1}{2}$ base $\times$ altitude.
6. Draw $\overline{OC} \perp \overline{AB}$.
7. In isosceles triangle AOB, $\overline{OC}$ bisects $\angle AOB$, making $m\angle AOC = 40$.
8. In rt. $\triangle AOC$,

$$\cos 40° = \frac{\text{length of adj. leg}}{\text{length of hypotenuse}}$$

$$\cos 40° = \frac{x}{6}$$

$$0.7660 = \frac{x}{6}$$

$$x = 6(0.7660)$$

$$x = 4.5960, \text{ or}$$

$$x = 4.6$$

In rt. $\triangle AOC$,

$$\sin 40° = \frac{\text{length of opp. leg}}{\text{length of hypotenuse}}$$

$$\sin 40° = \frac{y}{6}$$

$$0.6428 = \frac{y}{6}$$

$$y = 6(0.6428)$$

$$y = 3.8568, \text{ or}$$

$$y = 3.9$$

9. In isosceles triangle AOB, $\overline{OC}$ bisects base $\overline{AB}$. Therefore, AC, or $y = \frac{1}{2}AB$.
10. Area of $\triangle AOB = \frac{1}{2}$ base $\times$ altitude $= (y)(x) = (3.9)(4.6) = 17.94$, or 17.9.

NOTE. The area of $\triangle AOB$ can also be found by using the formula: area of $\triangle AOB = \frac{1}{2}AO \times OB \times \sin \angle AOB$, or $\frac{1}{2} \times 6 \times 6 \times \sin 80°$, etc.

11. Area of segment ASB = area of sector $OASB$ − area of $\triangle AOB$.
12. Area of segment $ASB = 25.1 - 17.9 = 7.2$, or 7.

Answer: Area of segment $ASB = 7$ sq. in.

EXERCISES

In 1 and 2, the answers may be left in radical form and in terms of π.

1. In a circle whose radius is 6, find the area of a minor segment whose arc has a central angle which contains:
a. $90°$ *b.* $60°$ *c.* $30°$ *d.* $120°$ *e.* $140°$

2. In a circle, a chord is drawn whose length is equal to the length of a radius. Find the area of the minor segment formed by the chord and its arc if the length of a radius of the circle is:
a. 6 *b.* 18 *c.* 4 *d.* 10 *e.* 5

3. [In this exercise, use $\pi = 3.14$, $\sqrt{3} = 1.73$, and express each result to the *nearest tenth.*] In circle O, chord $\overline{AB}$ and a radius are each 6 inches in length. If radii $\overline{OA}$ and $\overline{OB}$ are drawn, find: (*a*) the length of minor arc $\widehat{AB}$. (*b*) the area of triangle AOB. (*c*) the area of sector AOB. (*d*) the area of the minor segment of the circle.

4. A chord $\overline{AB}$ of circle O is 10 inches long and is 5 inches from the center of the circle. Radii $\overline{OA}$ and $\overline{OB}$ are drawn. Find: (*a*) the number of degrees contained in angle AOB. (*b*) the length of a radius of the circle. [Answer may be left in radical form.] (*c*) the area of triangle AOB. (*d*) the area of the minor sector of the circle. [Answer may be left in terms of π.] (*e*) the area of the minor segment of the circle. [Answer may be left in terms of π.] (*f*) the area of the major segment of the circle. [Answer may be left in radical form.]

5. The length of a radius of a circle is 12 and a minor segment of this circle has a chord whose length is equal to the length of the radius. (*a*) Find the perimeter of the minor segment. (*b*) Find the area of the minor segment. (*c*) Find the area of the major segment of the circle. [Answers may be left in radical form and in terms of π.]

6. A man has a flower garden in the shape of a minor segment of a circle. The arc of the segment contains $90°$ and the radius of the circle is 28 feet long. [Use $\pi = \frac{22}{7}$.] (*a*) Find to the *nearest foot* the number of feet of fencing required to enclose the garden. (*b*) Find the area of the garden.

7. In circle O, central angle AOB contains $90°$, the length of $\widehat{AB} = 6\pi$, and chord $\overline{AB}$ is drawn. (*a*) Find the length of radius $\overline{OA}$. (*b*) Find the area of triangle AOB. (*c*) Find the area of sector AOB. [Answer may be left in terms of π.] (*d*) Find the area of minor segment AB. [Answer may be left in terms of π.]

8. Find the sum of the areas of the three segments formed by inscribing an equilateral triangle in a circle whose radius is 24. [Answer may be left in terms of radicals and π.]

9. Find the sum of the areas of the four segments formed by inscribing a square in a circle whose radius is 8.

8. The Equilateral Triangle and Its Circles

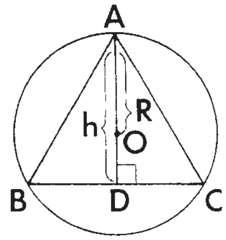


Fig. 8–19

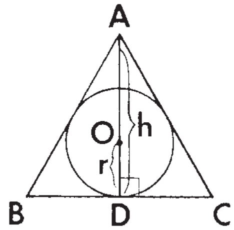


Fig. 8–20

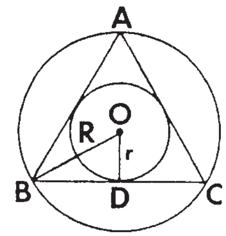


Fig. 8–21

Theorem 126. **The length of the radius of a circle circumscribed about an equilateral triangle is equal to two-thirds of the length of the altitude of the triangle.**

In Fig. 8–19, if circle O is circumscribed about equilateral triangle ABC, the length of the radius of the circle, OA, represented by R, is equal to $\frac{2}{3}$ of the length of the altitude of the triangle, AD, represented by h, or $R = \frac{2}{3}h$.

Theorem 127. **The length of the radius of a circle inscribed in an equilateral triangle is equal to one-third of the length of the altitude of the triangle.**

In Fig. 8–20, if circle O is inscribed in equilateral triangle ABC, the length of the radius of the circle, OD, represented by r, is equal to $\frac{1}{3}$ of the length of the altitude of the triangle, AD, represented by h, or $r = \frac{1}{3}h$.

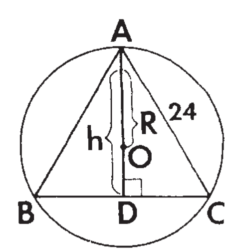
Corollary T126, 127–1. **The length of the radius of a circle circumscribed about an equilateral triangle is twice the length of the radius of the circle inscribed in the equilateral triangle.**

In Fig. 8–21, if the length of the radius of the circle circumscribed about equilateral triangle ABC is represented by R and the length of the radius of the circle inscribed in equilateral triangle ABC is represented by r, then $R = 2r$.

MODEL PROBLEM

Find, in radical form, the length of the radius of a circle circumscribed about an equilateral triangle the length of whose side is 24.

Solution:



1. In equilateral triangle ABC, draw $\overline{AD}$, the altitude to side $\overline{BC}$. Represent the length of $\overline{AD}$ by h.
2. $h = \frac{s}{2}\sqrt{3} = \frac{24}{2}\sqrt{3} = 12\sqrt{3}$. $\quad s = 24$.
3. $R = \frac{2}{3}h = \frac{2}{3}(12\sqrt{3}) = 8\sqrt{3}$.

Answer: Length of the radius is $8\sqrt{3}$.

EXERCISES

1. Find the length of the radius of the circle inscribed in an equilateral triangle the length of whose altitude is:
a. 12 *b.* 6 *c.* 15 *d.* 24 *e.* 7

2. Find the length of the radius of the circle circumscribed about an equilateral triangle the length of whose altitude is:
a. 6 *b.* 12 *c.* 18 *d.* 36 *e.* 5

3. Find the length of the radius of the circle inscribed in an equilateral triangle the length of whose side is:
a. 6 *b.* 12 *c.* 18 *d.* 9 *e.* 4

4. Find the length of the radius of the circle circumscribed about an equilateral triangle the length of whose side is:
a. 6 *b.* 12 *c.* 18 *d.* 9 *e.* 8

5. The length of the radius of a circle inscribed in an equilateral triangle is 5. Find the length of the radius of the circumscribed circle.

6. The length of the radius of a circle inscribed in an equilateral triangle is 6. Find the length of the altitude of the triangle.

7. The length of the radius of a circle circumscribed about an equilateral triangle is 12. Find the length of the altitude of the triangle.

8. The altitudes of an equilateral triangle ABC are concurrent in point O. If O is 4 inches from side $\overline{AB}$, how many inches is O from vertex A?

9. If the area of an equilateral triangle is $36\sqrt{3}$, find: (a) the length of a side of the triangle. (b) the length of an altitude of the triangle. (c) the circumference of the inscribed circle in terms of π. (d) the circumference of the circumscribed circle in terms of π. (e) the ratio of the circumferences of the two circles. (f) the area of the inscribed circle in terms of π. (g) the area of the circumscribed circle in terms of π. (h) the ratio of the areas of the two circles.

10. The area of a circle inscribed in an equilateral triangle is 16π. (a) Find the circumference of the circumscribed circle in terms of π. (b) Find the area of the circumscribed circle in terms of π. (c) The area of the circumscribed circle is how many times as large as the area of the inscribed circle?

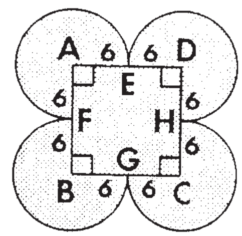
9. Solving More Difficult Area Problems

Sometimes the region whose area we are required to find is one for which there is no convenient area formula. In such a case, we try to combine regions for which there are convenient area formulas so that the result will be the area we are required to find. See how this is done in the following problems:

MODEL PROBLEMS

1. Find, in terms of π, the area of the shaded region.

Solution:



1. The shaded region consists of square $ABCD$, whose side is 12, and four major sectors whose areas are equal. In each sector, the central angle contains $(360° - 90°)$, or $270°$, and the length of the radius is 6.

2. Area of the square $= (AB)^2 = (12)^2 = 144$.

3. Area of each sector $= \frac{n}{360} \times \pi r^2 = \frac{270}{360} \times (6)^2\pi$
$= \frac{3}{4}(36\pi) = 27\pi$

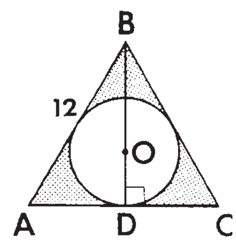
4. Area of shaded region = area of square + area of 4 sectors

5. Area of shaded region $= 144 + 4(27\pi)$

6. Area of shaded region $= 144 + 108\pi$

Answer: Area of shaded region $= 144 + 108\pi$.

2. A circle is inscribed in an equilateral triangle whose side is 12. Find to the *nearest integer* the difference between the area of the triangle and the area of the circle $[\pi = 3.14$ and $\sqrt{3} = 1.73]$



Solution:

1. Area of equilateral $\triangle ABC = \frac{s^2}{4}\sqrt{3} = \frac{(12)^2}{4}\sqrt{3}$
$= \frac{144}{4}\sqrt{3} = 36\sqrt{3}$
$= 36(1.73) = 62.28$,
or 62.3

2. In equilateral $\triangle ABC$, $BD = h = \frac{s}{2}\sqrt{3} = \frac{12}{2}\sqrt{3} = 6\sqrt{3}$.

3. Length of radius $\overline{OD} = r = \frac{1}{3}h = \frac{1}{3}(6\sqrt{3}) = 2\sqrt{3}$.

4. Area of circle $O = \pi r^2 = \pi(2\sqrt{3})^2 = 12\pi = 12(3.14) = 37.68$, or 37.7.

5. Required area = area of triangle ABC − area of circle O

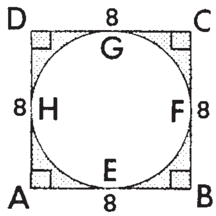
6. Required area $= 62.3 - 37.7 = 24.6$

7. Required area = 25 to the nearest integer

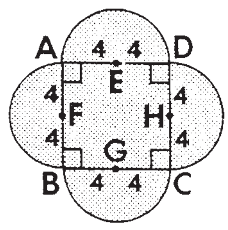
Answer: 25.

EXERCISES

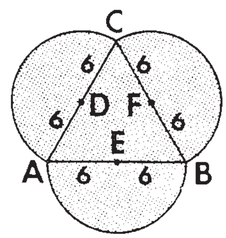
In 1–8, find the area of the shaded region. [Answers may be left in terms of radicals and π.]



1.



2.



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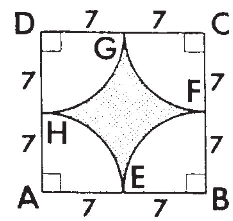

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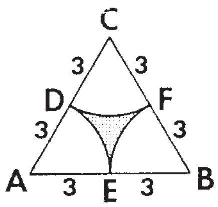

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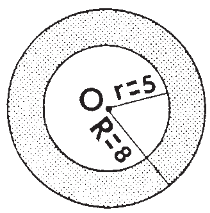

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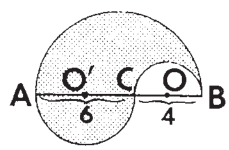

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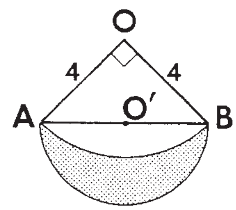
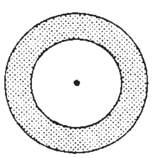


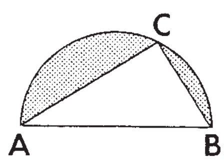
8.

9. From a square piece of tin, each of whose sides is 14 inches, is cut a circle whose diameter is 14 inches. Find to the *nearest square inch* the amount of tin that is wasted.

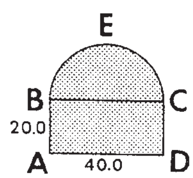
10. Nine circular discs, each 3 inches in diameter, were cut from a square of aluminum alloy 9 inches on a side. How much of the metal was wasted? Give your answer to the *nearest square inch.* [$\pi = \frac{22}{7}$]



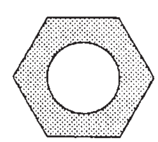
Ex. 11



Ex. 12



Ex. 13



Ex. 14

11. Find, in terms of π, the area of the circular ring (annulus) formed by two concentric circles, one with a radius of 7 and the other with a radius of 10.

12. In the semicircle, the length of chord $\overline{AC}$ is 16, and the length of chord $\overline{BC}$ is 12. Find the area of the shaded region to the *nearest integer.* [$\pi = 3.14$]

13. *ABECD* represents the cross section of an underground tunnel. *ABCD* is a rectangle 40.0 feet by 20.0 feet, surmounted by the semicircle *BEC*. Find to the *nearest square foot* the area of the cross section. [$\pi = 3.14$]

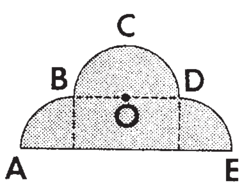
14. The figure represents the cross section of a hexagonal nut. Assuming that the diameter of the circle and the side of the regular hexagon are each 2 inches in length, find to the *nearest square inch* the area of the cross section (the shaded region).

15. A circle whose radius is 4 is inscribed in an equilateral triangle. (*a*) Find the length of the altitude of the triangle. (*b*) Find the length of the side of the triangle. [Answer may be left in radical form.] (*c*) Show that the difference between the area of the triangle and the area of the circle is approximately 33. [$\pi = 3.14$ and $\sqrt{3} = 1.73$]

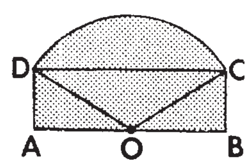
16. A circle and an equilateral triangle each have a perimeter of 132 feet. (*a*) Find the length of a side of the triangle. (*b*) Find the length of the radius of the circle. (*c*) Show that the difference between the area of the circle and the area of the triangle is approximately 549 square feet. [$\pi = \frac{22}{7}$ and $\sqrt{3} = 1.73$]

17. The area of a regular hexagon is $96\sqrt{3}$. Find the area of an equilateral triangle whose perimeter is equal to the perimeter of the hexagon.

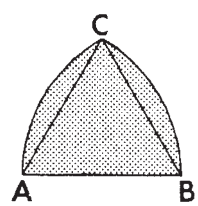
18. The radius of a circular flower bed is 30 feet and this bed is surrounded by a circular path 3 feet wide. Find the cost of paving the path at \$1.25 per square foot. [$\pi = \frac{22}{7}$]



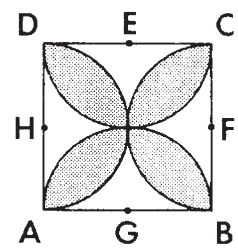
Ex. 19



Ex. 20



Ex. 21



Ex. 22

19. BCD is a semicircle and $\widehat{AB}$ and $\widehat{ED}$ are quadrants of congruent circles. (A quadrant is a quarter of a circle.) $\overline{AE}$ measures 56 inches and the diameter of the semicircle is 28 inches. Find the area of the entire region. [Use $\pi = \frac{22}{7}$.]

20. $ABCD$ is a rectangle. O is the midpoint of the longer side $\overline{AB}$. Arc $\widehat{DC}$ is an arc of the circle whose center is O. If $\overline{AD}$ measures 6 feet and angle DOC contains 120°, find the area of the entire region to the *nearest square foot*. [Use $\pi = 3.14$ and $\sqrt{3} = 1.73$.]

21. Arc $\widehat{CB}$ is the arc of a circle with A as the center, arc $\widehat{AC}$ is the arc of a circle with B as the center, and $AB = 24$.

a. Find the area of sector BAC. [Answer may be left in terms of π.]

b. Find the area of the segment bounded by chord $\overline{BC}$ and arc $\widehat{BC}$. [Answer may be left in radical form and in terms of π.]

c. Find the area of the entire region. [Answer may be left in radical form and in terms of π.]

22. $ABCD$ is a square each of whose sides is 6 inches in length. E, F, G, and H are the centers of semicircles which are constructed on the sides of the square, each side of the square being a diameter. Find to the *nearest square inch* the area of the shaded region.

10. Ratios of Circumferences of Circles

We have previously learned the following relationships:

The ratio of the circumferences of two circles is equal to the ratio of the lengths of their radii or the ratio of the lengths of their diameters:

$$\frac{C}{C'} = \frac{r}{r'} \quad \text{and} \quad \frac{C}{C'} = \frac{d}{d'}$$

Another way of stating these relationships is as follows:

The circumference of a circle varies directly as its radius or as its diameter.

This means that when the length of the radius or the diameter of a circle is multiplied by (or divided by) a positive number, the circumference is multiplied by (or divided by) the *same* number. If the length of the radius of a circle is multiplied by 5, the circumference of the circle is also multiplied by 5; if the length of the radius of a circle is divided by 2, the circumference of the circle is also divided by 2.

MODEL PROBLEMS

1. The lengths of the radii of two circles are 15 in. and 5 in. Find the ratio of the circumferences of the two circles.

Solution:

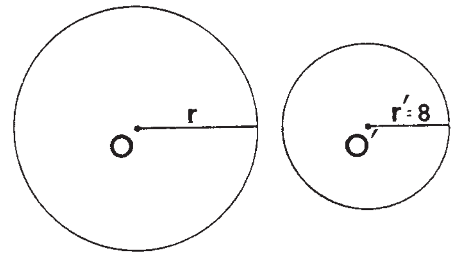
1. $\frac{C}{C'} = \frac{r}{r'}$ $\qquad r = 15$ in., $r' = 5$ in.

2. $\frac{C}{C'} = \frac{15}{5} = \frac{3}{1}$

Answer: Ratio of the circumferences is 3:1.

2. The ratio of the circumferences of two circles is 3:2 and the smaller circle has a radius of 8. Find the length of a radius of the larger circle.

Solution:



1. $\frac{C}{C'} = \frac{r}{r'}$ $\qquad \frac{C}{C'} = \frac{3}{2}$, $r' = 8$.

2. $\frac{3}{2} = \frac{r}{8}$

3. $2r = 24$

4. $r = 12$

Answer: Length of a radius of the larger circle is 12.

EXERCISES

1. Find the ratio of the circumferences of two circles if the ratio of the lengths of their radii is:
a. 3:1 *b.* 1:2 *c.* 5:4 *d.* $r{:}s$

2. Find the ratio of the circumferences of two circles if the ratio of the lengths of their diameters is:
a. 2:1 *b.* 1:4 *c.* 4:7 *d.* $m{:}n$

3. The diameters of two circles are 6 feet and 4 feet in length. Find the ratio of the circumferences of the circles.

4. The radii of two circles are 6 inches and 2 feet in length. Find the ratio of the circumferences of the circles.

5. The circumferences of two circles are in the ratio of 2:5. Find the ratio of the lengths of the radii of the circles.

6. State by what number the circumference of a circle is multiplied when the length of its radius is multiplied by:
a. 2 *b.* 5 *c.* 10 *d.* x

7. By what number must the circumference of a circle be multiplied in order to multiply the length of the diameter by 3?

8. The ratio of the circumferences of two circles is 5:4 and the larger circle has a radius whose length is 25. Find the length of the radius of the smaller circle.

9. The lengths of the radii of two circles are in the ratio of 4:1. If the circumference of the larger circle is 24π, find the circumference of the smaller circle.

10. The lengths of the diameters of two circles are in the ratio of 3:5. If the circumference of the smaller circle is 15π, find the circumference of the larger circle.

11. The ratio of the circumferences of two circles is 5:1. If the length of the radius of the larger circle exceeds the length of the radius of the smaller circle by 8, find the length of the radius of the smaller circle.

12. The lengths of the radii of two circles are in the ratio 2:1. If the circumference of the larger circle is 16π more than the circumference of the smaller circle, find the circumference of the larger circle.

13. What is the ratio of the circumference of the circle inscribed in an equilateral triangle to the circumference of the circle circumscribed about the equilateral triangle?

14. What is the ratio of the circumference of the circle circumscribed about a square to the circumference of the circle inscribed in the square?

11. Ratios of Areas of Circles

Theorem 128. **The ratio of the areas of two circles is equal to the ratio of the squares of the lengths of their radii, the squares of the lengths of their diameters, or the squares of their circumferences.**

In circles O and O' (Fig. 8–22), whose areas are represented by A and A', the lengths of whose radii are represented by r and r', the lengths of whose diameters are represented by d and d', and whose circumferences are represented by C and C':

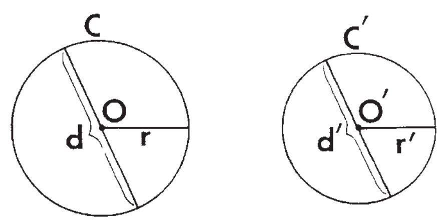


Fig. 8–22

$$\frac{A}{A'} = \frac{(r)^2}{(r')^2} \qquad \frac{A}{A'} = \frac{(d)^2}{(d')^2} \qquad \frac{A}{A'} = \frac{(C)^2}{(C')^2}$$

Another way of stating these relationships is as follows:

The area of a circle varies directly as the square of the radius, as the square of the diameter, or as the square of the circumference.

This means that when the length of the radius, the length of the diameter, or the circumference of a circle is multiplied by (or divided by) a positive number, the area is multiplied by (or divided by) the square of that number. If the length of the radius of a circle is multiplied by 3, the area of the circle is multiplied by the square of 3, that is, by 3^2, or 9; if the length of the diameter of a circle is divided by 4, the area of the circle is divided by the square of 4, that is, 4^2, or 16.

MODEL PROBLEMS

1. The lengths of the radii of two circles are in the ratio of 1:4. Find the ratio of the areas of the circles.

Solution:

1. $\frac{A}{A'} = \frac{(r)^2}{(r')^2} = \left(\frac{r}{r'}\right)^2 \qquad \frac{r}{r'} = \frac{1}{4}.$
2. $\frac{A}{A'} = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

Answer: 1:16.

2. The ratio of the areas of two circles is 9:4. The length of the radius of the larger circle is how many times the length of the radius of the smaller circle?

Solution:

1. $\frac{A}{A'} = \frac{(r)^2}{(r')^2} \qquad \frac{A}{A'} = \frac{9}{4}.$
2. $\frac{9}{4} = \frac{(r)^2}{(r')^2}$
3. $\frac{3}{2} = \frac{r}{r'}$ Positive square roots of equal quantities are equal.
4. Since the ratio of the length of the radius of the larger circle to the length of the radius of the smaller circle is $\frac{3}{2}$, the radius of the larger circle is $1\frac{1}{2}$ times as long as the radius of the smaller circle.

Answer: $1\frac{1}{2}$ times.

3. The ratio of the areas of two circles is 16:1. If the diameter of the smaller circle is 3, find the diameter of the larger circle.

Solution:

1. $\frac{A}{A'} = \frac{(d)^2}{(d')^2} \qquad \frac{A}{A'} = \frac{16}{1},\ d' = 3.$
2. $\frac{16}{1} = \frac{d^2}{(3)^2}$
3. $\frac{16}{1} = \frac{d^2}{9}$
4. $d^2 = 144$
5. $d = 12$

Answer: Diameter of the larger circle is 12.

EXERCISES

1. Find the ratio of the areas of two circles if the ratio of the lengths of their radii is:
a. 1:2 *b.* 4:1 *c.* 3:2 *d.* 4:25 *e.* 9:4 *f.* $a:b$

2. Find the ratio of the areas of two circles if the ratio of the lengths of their diameters is:
a. 1:5 *b.* 1:4 *c.* 1:9 *d.* 9:4 *e.* 9:16 *f.* $x:y$

3. The radii of two circles are 4 inches and 8 inches in length. Find the ratio of the areas of the circles.

4. The diameters of two circles are 8 inches and 1 foot in length. Find the ratio of the areas of the circles.

5. Find the ratio of the lengths of the radii of two circles if the ratio of their areas is:
a. 4:1 *b.* 1:9 *c.* 9:4 *d.* 9:16 *e.* 49:81 *f.* $a^2:b^2$

6. State by what number the area of a circle is multiplied when the length of its radius is multiplied by:
a. 2 *b.* 3 *c.* 4 *d.* 9 *e.* 10 *f.* m *g.* $2a$

7. State by what number the length of the radius of a circle must be multiplied in order to multiply the area by:
a. 4 *b.* 9 *c.* 1.44 *d.* $2\frac{1}{4}$ *e.* 2 *f.* x^2 *g.* x

8. If the length of the radius of a circle is increased by 100%, by how many per cent is the area increased?

9. If the length of the diameter of a large circular pipe is 3 times the length of the diameter of a small circular pipe, find the ratio of the area of the cross section of the small pipe to the area of the cross section of the large pipe.

10. The areas of two circles are in the ratio of 9:16. If the length of a radius of the large circle is 8, find the length of a radius of the small circle.

11. The lengths of the radii of two circles are in the ratio of 3:1. If the area of the smaller circle is 16π, find the area of the larger circle.

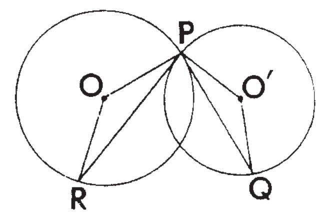
12. The area of a circle is 9 times the area of a smaller circle. If the length of a diameter of the larger circle exceeds the length of a diameter of the smaller circle by 8, find the length of a diameter of the smaller circle.

13. The lengths of the radii of two circles are in the ratio 2:1. If the area of the larger circle exceeds the area of the smaller circle by 75π, find the area of the smaller circle.

14. *a.* If the length of the radius of a circle is 16, find the length of an arc of this circle which contains 45°.
b. In another circle, an arc which contains 60° has the same length as the arc of 45° found in part *a*. Find the length of the radius of this other circle.

c. Find the ratio of the area of the smaller circle to the area of the larger circle.

15. In the figure, circles O and O', which are not congruent, intersect at P. Chord $\overline{PR}$ is tangent to circle O' and chord $\overline{PQ}$ is tangent to circle O. Radii $\overline{OP}$, $\overline{OR}$, $\overline{O'P}$, and $\overline{O'Q}$ are drawn.



Ex. 15

a. Prove that:
(1) $\angle OPR \cong \angle O'PQ$.
(2) $\triangle OPR \sim \triangle O'PQ$.

b. If the ratio of PR to PQ is $a{:}b$, find the ratio of the area of circle O to the area of circle O'.

16. Find the ratio of the area of a circle circumscribed about an equilateral triangle to the area of the circle inscribed in the equilateral triangle.

17. Find the ratio of the area of a circle circumscribed about a square to the area of the circle inscribed in the square.

12. Completion Exercises

Write a word or expression that, when inserted in the blank, will make the resulting statement true.

1. If a polygon is both equiangular and equilateral, it is a(an) ______ polygon.

2. A radius of a regular polygon is a radius of the ______ circle.

3. An apothem of a regular polygon is a radius of the ______ circle.

4. A circle may be circumscribed about any ______ polygon.

5. The apothem of a regular polygon is ______ to the side to which it is drawn.

6. A regular polygon of four sides is called a(an) ______.

7. Regular polygons of the same number of sides are ______.

8. The area of a regular octagon the length of whose side is represented by s and the length of whose apothem is represented by a is represented by ______.

9. If an exterior angle of a regular polygon has the same measure as an interior angle, the polygon is a(an) ______.

10. The number π is a constant which represents the ratio of the circumference of a circle to the ______ of the circle.

11. A side of a regular hexagon inscribed in a circle is congruent to a(an) ______ of the circle.

12. The circumference of a circle is equal to π times the ______ of the circle.

13. The area of a circle is equal to π times the ________ of the length of the radius of the circle.
14. A region bounded by an arc of a circle and its chord is the interior region of a(an) ________ of the circle.
15. A region bounded by two radii of a circle and the arc which they intercept on the circle is the interior region of a(an) ________ of the circle.
16. The length of a radius of a circle inscribed in an equilateral triangle is equal to ________ of the length of an altitude of the triangle.
17. The length of a radius of a circle circumscribed about an equilateral triangle is ________ times the length of a radius of the inscribed circle.
18. The ratio of the ________ of two circles is the same as the ratio of the lengths of their radii.
19. The ratio of the areas of two circles is the same as the ratio of the ________ of their radii.
20. The regular polygon the length of whose apothem is one-half the length of its side is a(an) ________.
21. If the length of a radius of a circle is multiplied by 2, the circumference of the circle is multiplied by ________.
22. If the length of a radius of a circle is multiplied by 2, the area of the circle is multiplied by ________.
23. In order to multiply the circumference of a circle by 9, the length of the radius of the circle must be multiplied by ________.
24. In order to multiply the area of a circle by 9, the length of a radius of the circle must be multiplied by ________.
25. In order to multiply the area of a circle by 3, the length of a diameter of the circle must be multiplied by ________.

13. True-False Exercises

If the statement is always true, write *true;* if the statement is not always true, write *false.*

1. Polygons that are equiangular must be equilateral.
2. Regular polygons of the same number of sides are similar.
3. Sectors of circles that are not congruent may be congruent.
4. The length of a radius of a circle inscribed in a triangle is one-third the length of an altitude.
5. An apothem of a regular polygon is a radius of the circumscribed circle.
6. The diagonals of a regular polygon must be congruent.
7. The area of a regular polygon is equal to one-half the product of its perimeter and the length of its apothem.
8. A radius of a regular polygon bisects the angle to whose vertex it is drawn.

9. The length of an apothem of a square equals one-half the length of a side of the square.
10. If the radius of a circle is 2 inches, the number of square inches in the area of the circle is the same as the number of inches in its circumference.
11. If the length of a radius of a circle is multiplied by a positive number s, the circumference of the circle is multiplied by s.
12. If the length of a radius of a circle is multiplied by d, a positive number other than 1, the area of the circle is multiplied by d.
13. The ratio of the circumference of a circle to the length of its diameter is a constant.
14. A rhombus is a regular polygon.
15. The ratio of the length of a radius of the circle circumscribed about an equilateral triangle to the length of a radius of the inscribed circle is 2:1.
16. As the number of sides of a regular polygon inscribed in a circle increases, the length of the apothem of the polygon decreases.
17. If an equilateral polygon is inscribed in a circle, the segments drawn from the center of the circle perpendicular to the sides are congruent.
18. A central angle of a regular polygon is supplementary to an interior angle of the polygon.
19. As the number of sides of a regular polygon increases, the number of degrees contained in a central angle of the polygon increases.
20. It is possible for an exterior angle of a regular polygon to contain 70°.

14. "Always, Sometimes, Never" Exercises

If the blank space in each of the following exercises is replaced by the word *always*, *sometimes*, or *never*, the resulting statement will be true. Select the word which will correctly complete each statement.

1. An equilateral polygon is ______ a regular polygon.
2. A regular polygon is ______ equiangular.
3. An equilateral polygon inscribed in a circle is ______ regular.
4. Two polygons are ______ congruent if their corresponding sides are congruent.
5. If the length of a radius of a circle is multiplied by a positive number k, the circumference is ______ multiplied by k.
6. The ratio of the areas of two regular polygons is ______ equal to the ratio of the squares of the lengths of their sides.
7. A circle can ______ be circumscribed about any regular polygon.
8. If a polygon is equilateral, it is ______ equiangular.

9. Sectors whose central angles contain the same number of degrees and which are drawn in circles that are not congruent are ______ congruent.
10. As the number of sides of a regular polygon inscribed in a circle increases, the length of its apothem ______ increases.
11. The diagonals of a regular polygon are ______ congruent.
12. The length of an apothem of a regular polygon is ______ one-half the length of a side of the polygon.
13. The length of a radius of a regular polygon is ______ twice the length of an apothem of the polygon.
14. An equiangular polygon inscribed in a circle is ______ a regular polygon.
15. An interior angle of a regular polygon is ______ supplementary to a central angle of the polygon.
16. Regular polygons of the same number of sides are ______ similar.
17. A radius of a regular polygon ______ bisects the angle to whose vertex it is drawn.
18. The ratio of the areas of two circles that are not congruent is ______ the same as the ratio of the lengths of the radii of the two circles.
19. If the line segments which are drawn from the center of a circle perpendicular to and ending in the sides of an inscribed polygon are congruent, the polygon is ______ equilateral.
20. A regular polygon of n sides is inscribed in a circle. In the same circle a regular polygon of $2n$ sides is inscribed. The length of a side of the new polygon is ______ one-half the length of a side of the original polygon.
21. The area of an equilateral triangle inscribed in a circle ______ exceeds the area of a square inscribed in that circle.
22. The area of a regular polygon inscribed in a circle ______ exceeds the area of the circle.
23. If a circle is divided into three or more congruent arcs, the chords of these arcs ______ form a regular polygon.
24. A central angle of a regular polygon is ______ congruent to an exterior angle of the polygon.
25. The length of a side of a regular hexagon inscribed in a circle is ______ equal to the length of a radius of the circle.

15. Multiple-Choice Exercises

Write the letter preceding the word or expression that best completes the statement.

1. A circle can always be circumscribed about (*a*) an equiangular polygon (*b*) an equilateral polygon (*c*) a regular polygon.
2. The regular quadrilateral is a (*a*) rectangle (*b*) square (*c*) rhombus.

3. A regular decagon has a side whose length is represented by s and an apothem whose length is represented by a. The area of the decagon is represented by (*a*) $20as$ (*b*) $10as$ (*c*) $5as$.
4. Regular polygons of the same number of sides are always (*a*) equal in area (*b*) congruent (*c*) similar.
5. As the length of a radius of a circle increases, the ratio of the circumference to the length of a diameter of the circle (*a*) is constant (*b*) increases (*c*) decreases.
6. If the circumference of a circle is represented by C, the length of a radius of the circle is represented by (*a*) $\frac{C}{\pi}$ (*b*) $\frac{C}{2\pi}$ (*c*) $\sqrt{\frac{C}{\pi}}$.
7. If the length of a radius of a circle is increased by x, the circumference of the circle is increased by (*a*) x (*b*) $2x$ (*c*) $2\pi x$.
8. A regular polygon is defined as a polygon whose (*a*) angles are congruent (*b*) sides are congruent (*c*) sides are congruent and whose angles are congruent.
9. The ratio of the areas of any two regular polygons which have equal perimeters is equal to the ratio of (*a*) the lengths of their apothems (*b*) the squares of the lengths of their apothems (*c*) their perimeters.
10. The ratio of the areas of two circles is equal to the ratio of (*a*) the lengths of their radii (*b*) the square roots of the lengths of their radii (*c*) the squares of the lengths of their radii.
11. In a circle of radius r, the area of a sector whose central angle contains $n°$ is represented by (*a*) $\frac{n}{360} \times \pi r^2$ (*b*) $\frac{n}{180} \times \pi r^2$ (*c*) $\frac{n}{360} \times \pi d^2$.
12. The degree measure of a central angle of a regular polygon of n sides is represented by (*a*) $\frac{360}{n}$ (*b*) $\frac{180}{n}$ (*c*) $\frac{n}{360}$.
13. If the length of a radius of a circle is multiplied by 4, the area of the circle is multiplied by (*a*) 16 (*b*) 4 (*c*) 2.
14. If the circumference of a circle is multiplied by 9, the length of a radius of the circle is multiplied by (*a*) 3 (*b*) 9 (*c*) 81.
15. Which conclusion makes the following statement false? If the number of sides of a regular polygon inscribed in a circle is increased (*a*) the length of an apothem decreases (*b*) the measure of each interior angle increases (*c*) the perimeter increases.
16. As the number of sides of a regular polygon inscribed in a given circle increases (*a*) the measure of the central angle increases (*b*) the length of an apothem increases (*c*) the measure of each exterior angle increases.
17. A central angle of a regular polygon and an interior angle of the polygon are always (*a*) supplementary (*b*) complementary (*c*) congruent.

18. If a regular polygon has diagonals which are not congruent, it may be (*a*) a square (*b*) a hexagon (*c*) a pentagon.

19. The area of a circle the length of whose diameter is represented by d is represented by (*a*) $\frac{1}{4}\pi d^2$ (*b*) $\frac{1}{2}\pi d^2$ (*c*) πd^2.

20. A central angle of a regular polygon and an exterior angle drawn at one of the vertices of the polygon are always (*a*) complementary (*b*) supplementary (*c*) congruent.

16. Construction Exercises

1. Inscribe a square in a given circle.

2. Inscribe a regular octagon in a given circle.

3. Inscribe a regular hexagon in a given circle.

4. Inscribe an equilateral triangle in a given circle.

5. Circumscribe a square about a given circle.

6. Circumscribe an equilateral triangle about a given circle.

7. Using a given line segment as a side, construct a regular hexagon.

8. Construct a circle whose circumference is equal to the sum of the circumferences of two given circles whose radii are R and r.

9. Construct a circle whose circumference is equal to the difference of the circumferences of two given circles whose radii are R and r.

10. Construct a circle whose circumference is twice the circumference of a given circle whose radius is R.

11. Construct a circle whose area is equal to the sum of the areas of two given circles whose radii are R and r.

12. Construct a circle whose area is equal to the difference of the areas of two given circles whose radii are R and r.

13. Construct a circle whose area is twice the area of a given circle whose radius is R.

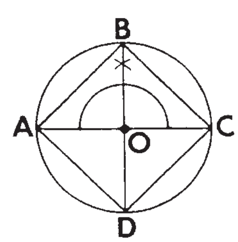
14. Construct a circle whose area is 4 times the area of a given circle whose radius is R.

15. Construct a circle whose area is one-quarter the area of a given circle whose radius is R.

16. The diagram shows the construction for inscribing a square in a given circle. Which statement, *a* or *b*, is used to prove the construction correct?

a. An equiangular polygon inscribed in a circle is a regular polygon.

b. If a circle is divided into any number of congruent arcs, the chords of these arcs form a regular inscribed polygon.



Ex. 16